

Lecture 10 - A

§6.1 Splines

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We will answer these questions:

- What are splines? In what way are they advantageous over standard polynomial interpolation?
- What makes a spline different from a piecewise polynomial?
- Is a degree 1 spline unique?
- What are the conditions on a degree 2 spline? Is it unique?



Problems with Polynomial Interpolation

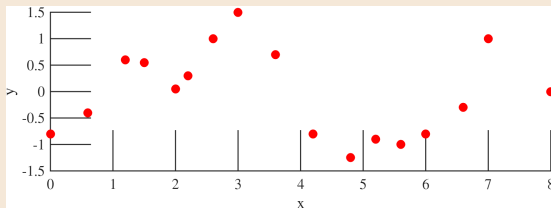


Runge's Phenomenon

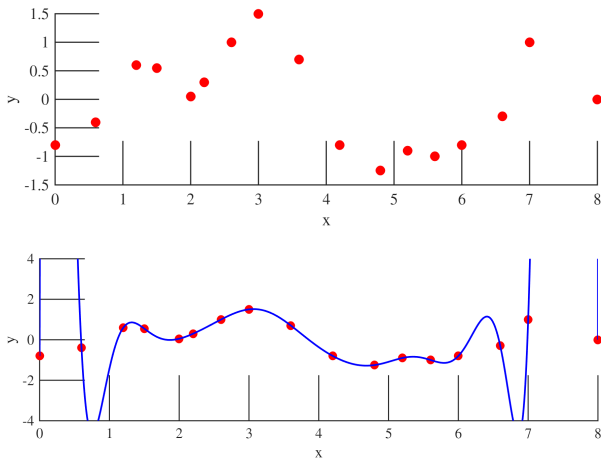
Runge's Phenomenon

- Suppose we interpolate large data set using **a single high-degree polynomial**
- Polynomial interpolation at equally spaced points can exhibit large oscillations at the boundaries

Example - Try interpolation on this data set



Runge's Phenomenon



Key Idea

Fitting a high-degree polynomial on a data set can lead to disappointing results.

Non-differentiable Functions

- Real data is often:., noisy, piecewise-defined, or non-smooth
- But a polynomial is infinitely differentiable
- This forces **artificial smoothness** everywhere

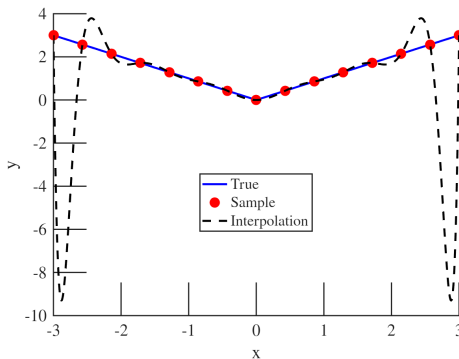


Figure: Interpolation on $f(x) = |x|$ has large errors at the boundaries



Instability

- Polynomial interpolation is a **global** method.
- Changing one data point affects the entire polynomial.

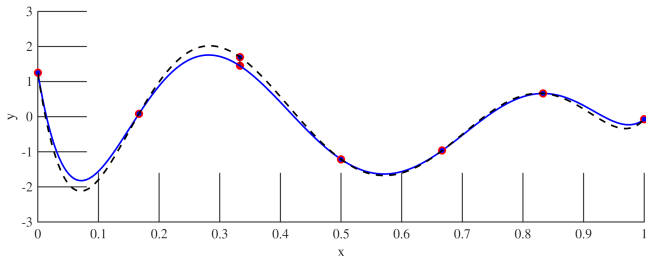


Figure: A local change in data causes global error

Instability

A small local perturbation affects the entire interpolation.

The Solution

What we tried

Create one big polynomial $p(x)$ to fit the data (x_i, y_i) .

Issues

- Polynomial degree grows with number of data points
- Runge's phenomenon
- Not good for non-differentiable functions
- Local change in data causes global changes



The Solution

What we tried

Create one big polynomial $p(x)$ to fit the data (x_i, y_i) .

Issues

- Polynomial degree grows with number of data points
- Runge's phenomenon
- Not good for non-differentiable functions
- Local change in data causes global changes

The Fix

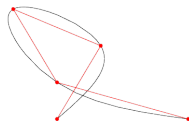
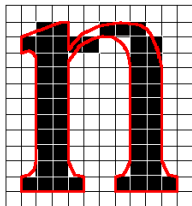
splines - create multiple little polynomials



Splines



Why Splines?



- Used for smooth lines and surfaces
- Truetype fonts, smooth textures
- Use low-degree **piecewise polynomials**

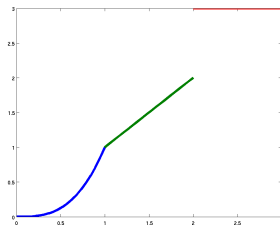


Piecewise Polynomial

A function $f(x)$ is a **piecewise polynomial** on $[a, b]$ if there exists a (finite) partition P of $[a, b]$ such that $f(x)$ is a polynomial on each $[t_i, t_{i+1}] \in P$.

Example

$$f(x) = \begin{cases} x^3 & x \in [0, 1] \\ x & x \in (1, 2) \\ 3 & x \in [2, 3] \end{cases}$$



Piecewise Polynomial vs. Spline

Question

What makes a spline different from a piecewise function?

Definition: Spline

A **spline function** $S(x)$ is a piecewise function consisting of polynomials joined together with certain "smoothness" conditions.

$$S(x) = \begin{cases} S_0(x), & x \in [t_0, t_1] \\ S_1(x), & x \in [t_1, t_2] \\ \vdots & \vdots \\ S_{n-1}(x), & x \in [t_{n-1}, t_n] \end{cases}$$

- Splines consist of polynomials of the same degree
- Must interpolate a data set (x_i, y_i)
- Must satisfy a global "smoothness" condition



Notations

Data Set

Use t and y for the given data set

$$(t_i, y_i), \quad \text{for } i = 0, 1, 2, \dots, n$$

Objective

Construct spline function $S(x)$ such that

$$S(t_i) = y_i$$

and is composed of polynomials

$$S_i(x), \quad \text{for } x \in [t_i, t_{i+1})$$

Method: Spline Interpolation

- **Given:** Set of points (t_i, y_i) , Degree of smoothness.
- **Output:** Continuous piecewise polynomial $S(x)$
- **Idea:** Degree of the polynomial is fixed, yet we can still satisfy $S(t_i) = y_i$ for all points



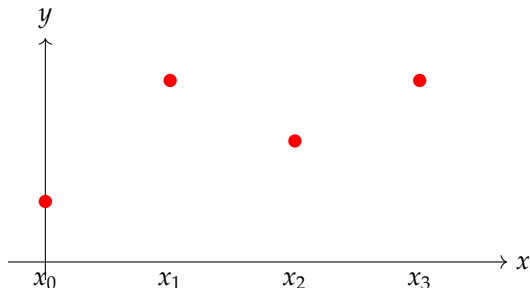
Degree One Spline



The Simplest Piecewise Interpolant

Question

What is the simplest piecewise function that interpolates these data?



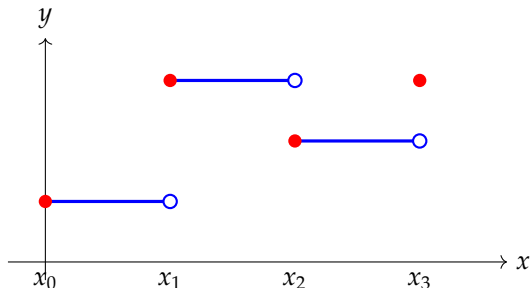
- This is called a **Degree Zero Spline**



The Simplest Piecewise Interpolant

Question

What is the simplest piecewise function that interpolates these data?

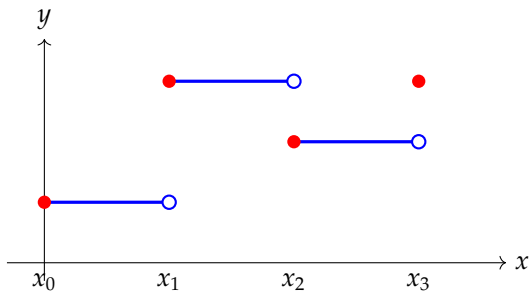


- This is called a **Degree Zero Spline**



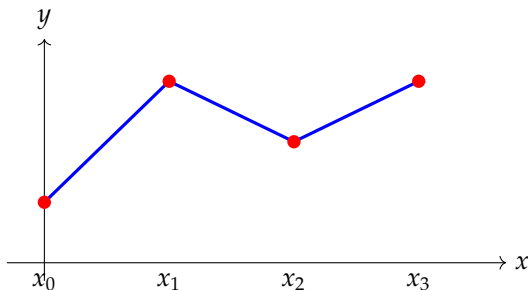
What's the next simplest idea?

- Degree Zero splines has **jumps**.



What's the next simplest idea?

- Degree Zero splines has **jumps**.



Degree One: Piecewise Linear

Connect consecutive data points with straight lines.

- Interpolates the data
- All pieces are polynomials
- Globally continuous

First-Degree Spline

Definition: First-Degree Spline

A function $S(x)$ is a **spline of degree 1** if it satisfies all of the following:

- 1 The domain of $S(x)$ is an interval $[a, b]$
- 2 $S(x)$ is continuous on $[a, b]$
- 3 There is a partition $a = t_0 < t_1 < \dots < t_n = b$ such that $S(x)$ is linear on each subinterval $[t_i, t_{i+1}]$.

- First-degree spline is made up of lines that are connected end-to-end

Example

Is this function a spline of degree 1?

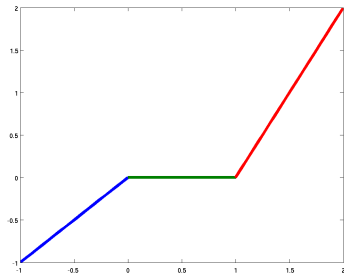
$$f(x) = \begin{cases} x & x \in [-1, 0] \\ 0 & x \in (0, 1) \\ 2x - 2 & x \in [1, 2] \end{cases}$$

Degree 1 Spline

Example

$$f(x) = \begin{cases} x & x \in [-1, 0] \\ 0 & x \in (0, 1) \\ 2x - 2 & x \in [1, 2] \end{cases}$$

- Is it continuous at $x = 0$? yes
- Is it continuous at $x = 1$? yes
- Is each component a line? yes
- Therefore $f(x)$ a first-degree spline



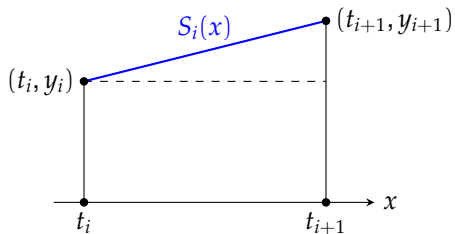
Formula for $S_i(x)$

There are three conditions on S_i

- 1 $S_i(t_i) = y_i$
- 2 $S_i(t_{i+1}) = y_{i+1}$
- 3 $S_i(x) = m(x - x_0) + y_0$

Write $S_i(x)$ in point-slope form

$$\begin{aligned} S_i(x) &= y_0 + m(x - x_0) \\ &= y_i + \frac{y_{i+1} - y_i}{t_{i+1} - t_i}(x - t_i) \end{aligned}$$



S



Example: Degree-1 Spline

Example

Construct the **degree-1 spline** that interpolates the data

$$(0, 1), \quad (1, 3), \quad (3, 2).$$

- On the interval $[0, 1]$

$$m_0 = \frac{3 - 1}{1 - 0} = 2$$

Thus,

$$S_0(x) = 1 + 2x, \quad 0 \leq x \leq 1.$$

- On the interval $[1, 3]$ Slope:

$$m_1 = \frac{2 - 3}{3 - 1} = -\frac{1}{2}$$

Thus,

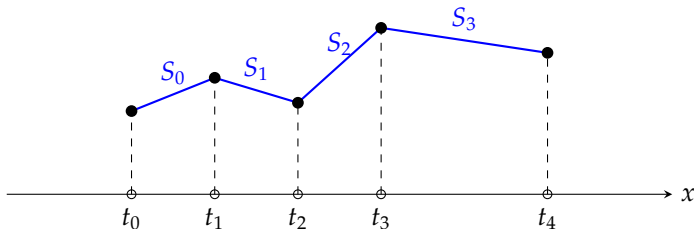
$$S_1(x) = 3 - \frac{1}{2}(x - 1), \quad 1 \leq x \leq 3.$$



Constraints vs. Unknowns



Number of Unknowns



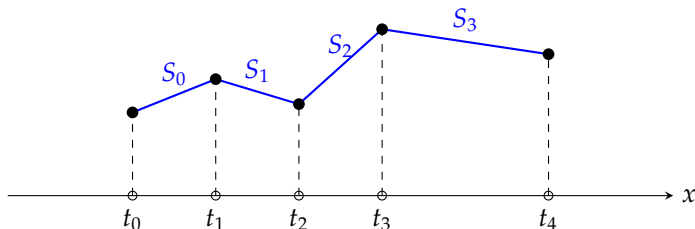
- Suppose there are 5 nodes (t_i, y_i) for $i = 0, 1, 2, 3, 4$
- Then there are $n = 4$ subintervals

Question

How many unknown coefficients are needed to specify the spline $S(x)$?



Number of Unknowns



- Each component has the form

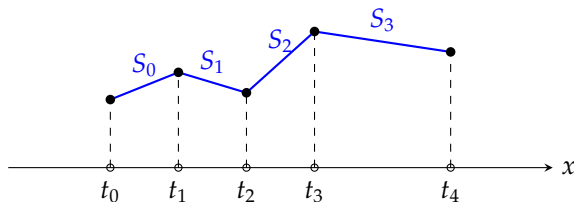
$$S_i(x) = m_i x + b_i, \quad i = 1, 2, 3, 4$$

- Therefore there are $4 \times 2 = 8$ unknowns

General Number of Unknowns

In general, a first-degree spline with $n + 1$ nodes will require computing $2n$ unknown coefficients.

Number of Constraints



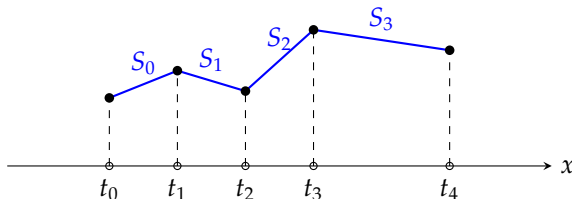
Question

How many constraints or equations are enforced on the spline $S(x)$ for the function above?

- Constraint on $S_0(x)$: $S_0(t_0) = y_0$ and $S_0(t_1) = y_1$
- Constraint on $S_1(x)$: $S_1(t_1) = y_1$ and $S_1(t_2) = y_2$
- Constraint on $S_2(x)$: $S_2(t_2) = y_2$ and $S_2(t_3) = y_3$
- Constraint on $S_3(x)$: $S_3(t_3) = y_3$ and $S_3(t_4) = y_4$



Number of Constraints



- For each of the $n = 4$ components $S_i(x)$, we have *two* constraints
- Therefore there are $4 \times 2 = \mathbf{8 \text{ constraints}}$

General Number of Unknowns

In general, a first-degree spline with $n + 1$ nodes will have $2n$ number of constraints.



Summary for Degree 1 Splines

Counting Unknowns

- There are n subintervals and each piece is linear:

$$S_i(x) = m_i x + b_i$$

- This gives $2n$ **unknown coefficients**

Counting Constraints

- Each spline piece must interpolate its two endpoints
- This enforces $2n$ **equations**

Key Idea

Since we have

Number of unknowns = number of constraints

the first-degree spline $S(s)$ is unique.

Continuity Condition

Definition: First-Degree Spline

A function $S(x)$ is a **spline of degree 1** if it satisfies all of the following:

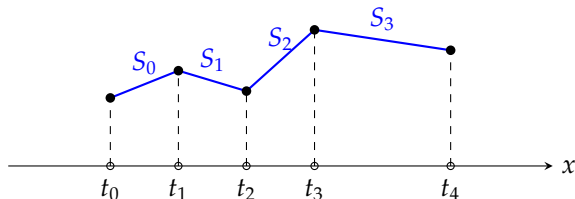
- ① The domain of $S(x)$ is an interval $[a, b]$
 - ② $S(x)$ **is continuous on** $[a, b]$
 - ③ Each piece $S_i(x)$ is linear
- The second property is the **continuity condition**
 - It is the bare minimum requirement for smoothness

Key Idea

If there are $n + 1$ nodes, then the continuity condition requires $2n$ constraints.



Smoothness



- The issue with a first degree polynomial is the *smoothness*
- Splines have applications in graphics

Key Idea

We need to increase the degree of the polynomial for visual smoothness.



Second-Degree Spline



Second-Degree Spline

Definition: Second-Degree Spline

A function $S(x)$ is a **spline of degree 2** if all conditions are satisfied:

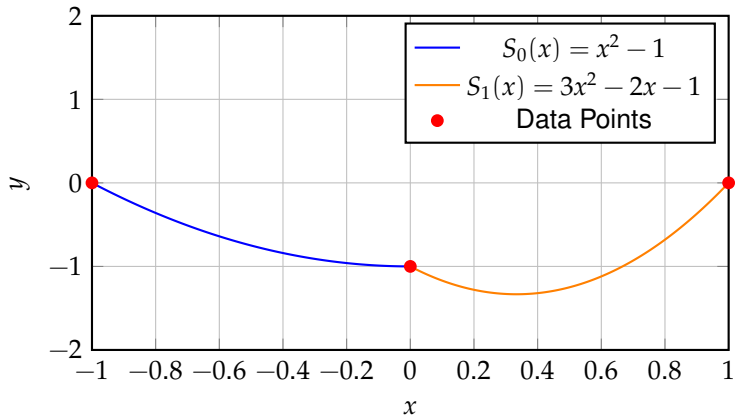
- 1 The domain of $S(x)$ is an interval $[a, b]$
- 2 $S(x)$ is continuous on $[a, b]$
- 3 $S'(x)$ is continuous on $[a, b]$
- 4 There is a partition $a = t_0 < t_1 < \cdots < t_n = b$ such that $S(x)$ is quadratic on each subinterval $[t_i, t_{i+1}]$.

Second degree splines have the form

$$S(x) = \begin{cases} S_0(x) & x \in [t_0, t_1] \\ S_1(x) & x \in [t_1, t_2] \\ \vdots & \vdots \\ S_{n-1}(x) & x \in [t_{n-1}, t_n] \end{cases}, \quad S_i(x) = a_i x^2 + b_i x + c_i$$



Is this a second-degree spline?

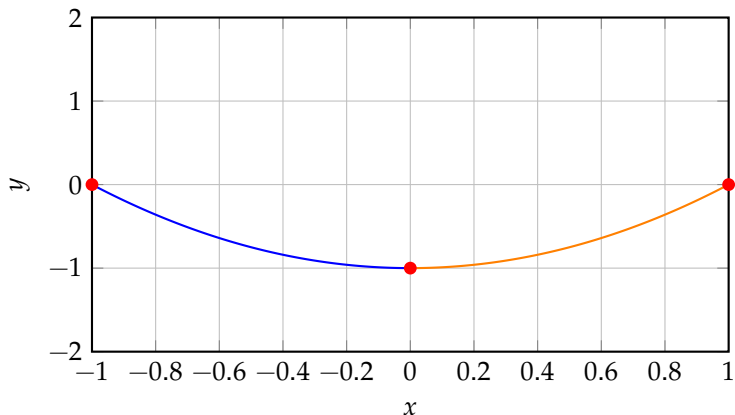


- Each component is a quadratic ✓
- $S(x)$ is continuous ✓

Question

What's the problem here?

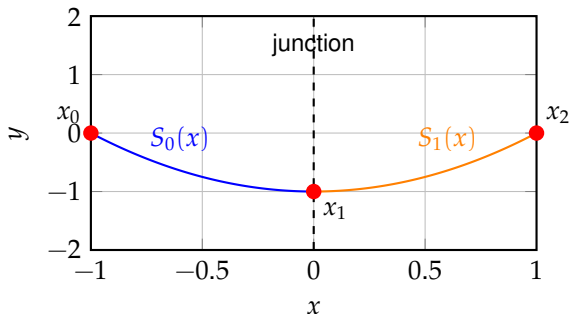
Smoothness



Key Idea

Unlike degree 1 splines, degree 2 splines must be smooth

Smoothness Condition

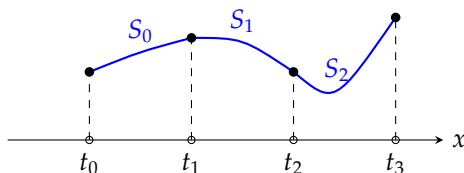


- The node x_1 separates the spline pieces $S_0(x)$ and $S_1(x)$
- To enforce smoothness, we require matching derivatives

$$S'_0(x_1) = S'_1(x_1)$$



Number of Unknowns



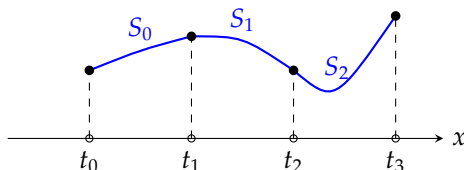
- Since each component has the form $S_i(x) = a_i x^2 + b_i x + c_i$, each have 3 unknown coefficients
- If there are n of them then there are

$$3 \times 3 = \mathbf{9 \text{ unknowns}}$$

for a spline of degree 2



Number of Constraints



- For each component $S_i(x)$ must satisfy endpoints so there are

$$3 \times 2 = \mathbf{6 \text{ constraints}}$$

- We require smoothness at the **two interior nodes** where quadratics meet

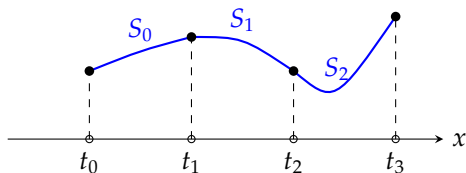
$$S'_1(t_1) = S'_2(t_1), \quad S'_1(t_2) = S'_2(t_2)$$

- Total constraints is

$$6 + 2 = \mathbf{8 \text{ constraints}}$$



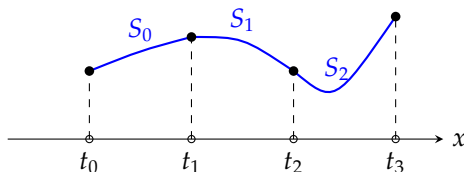
Constraints vs. Unknowns



- Number of unknowns: 9
- Number of constraints: 8



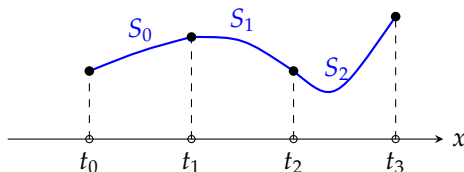
Constraints vs. Unknowns



- Number of unknowns: 9
- Number of constraints: 8
- **We are one constraint short**



Constraints vs. Unknowns



- Number of unknowns: 9
- Number of constraints: 8
- **We are one constraint short**
- Just make one up!

$$S_0''(t_0) = 0$$



General Second-Degree Splines

Counting Unknowns

- There are n subintervals and each piece is quadratic:

$$S_i(x) = a_i x^2 + b_i x + c_i$$

- This gives $3n$ **unknown coefficients**

Counting Constraints

- Components must interpolate two endpoints which gives $2n$ **equations**
- Smoothness at the interior points gives us $n - 1$ **equations**
- Total is $3n - 1$ **constraints**

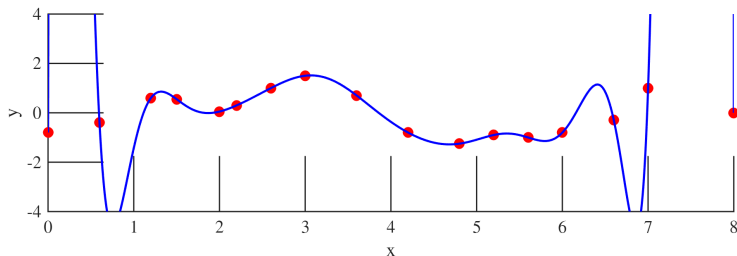
Key Idea

- Second-degree splines are **NOT** unique
- We need **ONE** extra condition such as $S'_0(t_0) = 0$ or $S''_0(t_0) = 0$

Summary

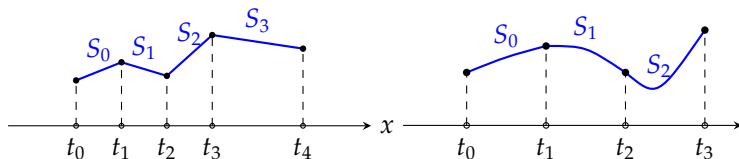


What is Runge's phenomenon?



- **Runge's phenomenon** is a notorious polynomial interpolation issue
- There are large fluctuations at the endpoints

What are Splines?



- Splines are piecewise polynomial interpolations with smoothness conditions
- Each spline piece is a low-degree polynomial

Key Idea

Instead of one big high-degree polynomial, use multiple low-order polynomials.

Counting Unknowns

- Suppose there are $n + 1$ data points. Then you have n spline pieces.
- Degree 1 Splines contain $2n$ unknowns because each piece is

$$S_i(x) = a_i x + b_i$$

- Degree 2 Splines contain $3n$ unknowns because each piece is

$$S_i(x) = a_i x^2 + b_i x + c_i$$



Uniqueness

- Degree 1 splines require
 - continuity: $2n$ conditions
- Degree 2 Splines require
 - continuity: $2n$ conditions
 - smoothness: $n - 1$ conditions
 - total: $3n - 1$ conditions

Key Idea

- There is one unique degree 1 spline
- Degree 2 splines are not unique because they are missing a condition



Moving Forward

- How to construct degree 2 splines by hand
- How to construct degree 2 splines efficiently
- Degree 3 splines
- Cost of constructing degree splines

