

Lecture 09 - A

§4.1 Polynomial Interpolation

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We will answer these questions:

- What is numerical interpolation supposed to accomplish?
- What is a monomial basis and what are its drawbacks?
- What is the Lagrange basis and why is it better?
- What is the Newton basis and why is it better than both the monomial and Lagrange?



Math Joke

What is the next number in the sequence?

1, 2, 3, ____



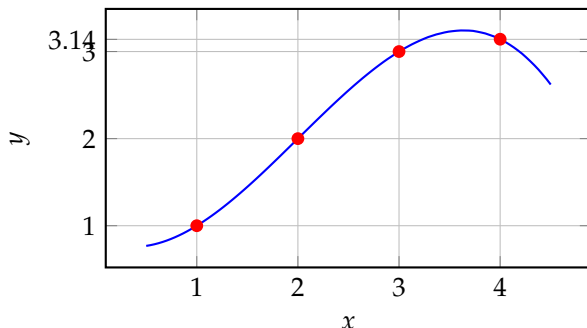
Math Joke

What is the next number in the sequence?

1, 2, 3, ____

It's π ! Plug in 1, 2, 3, and then 4 into the polynomial

$$p(x) = \frac{\pi x^3}{6} - \frac{2x^3}{3} - \pi x^2 + 4x^2 + \frac{11\pi x}{6} - \frac{19x}{3} - \pi + 4$$



Data Points



Working with Data Points

Objective

Given a set of data points, find an approximating function $g(x)$ to represent the data.

Why?

- Data analysis
- Prediction
- Scientific modeling

Types of approximating functions:

- 1 Polynomials
- 2 Piecewise polynomials
- 3 Rational functions
- 4 Trig functions



Curve Fitting

- The goal of **Curve fitting** is to find a continuous function that represents discrete data.
- Interpolation and Regression are forms of curve fitting

Interpolation

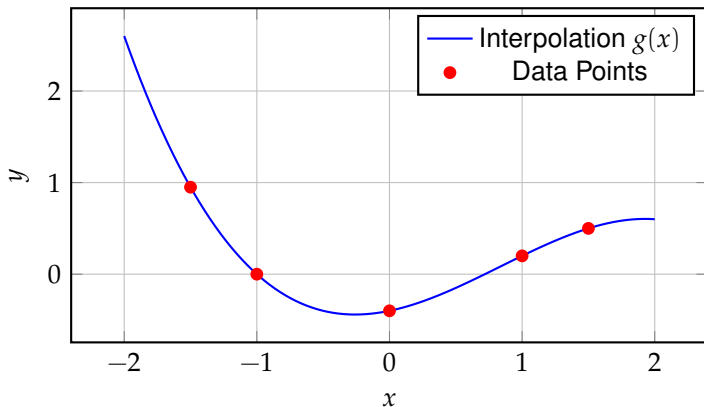
Construct a function $f(x)$ that cuts through all data points.

Regression

Find parameters for a function $f_p(x)$ that has a low over-all error at data points.



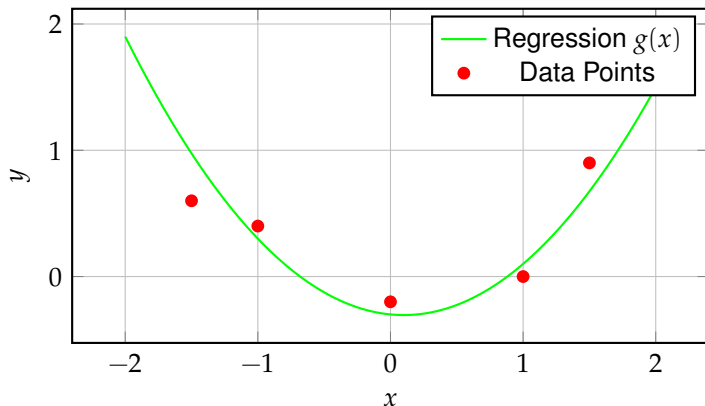
Interpolation



Interpolation: function cuts through all data points



Regression



Regression: find parameters for best-fitting parabola.

(We will cover this in Lecture 14)



Interpolation



Solvability of a Problem

- A problem in math may have N **unknowns** to solve for.
- You may be given M **conditions** or equations.

Rule of Thumb

- If $M > N$, you usually don't have a solution. (overly strict conditions)
- If $M < N$, you usually have infinite solutions. (too much freedom)
- If $M = N$, you may have a unique solution. (just right)

Definition: Linear Problems

- When $M = N$, we say the problem is **well-defined**.
- When $M > N$, we say the problem is **over-determined**.
- When $M < N$, we say the problem is **under-determined**.



Interpolation

Objective

Given $n + 1$ distinct points $x_0, \dots, x_n$, and values $y_0, \dots, y_n$, find a polynomial $p(x)$ of degree n so that

$$p(x_i) = y_i \quad i = 0, \dots, n$$

- A polynomial of degree n has $n + 1$ degrees-of-freedom:

$$p(x) = a_0 + a_1x + \dots + a_nx^n$$

- $n + 1$ constraints determine the polynomial uniquely:

$$p(x_i) = y_i, \quad i = 0, \dots, n$$

Key Idea

Interpolation means finding a polynomial that cuts through all data points.

Well-defined Problem

Theorem (page 157 7thEd)

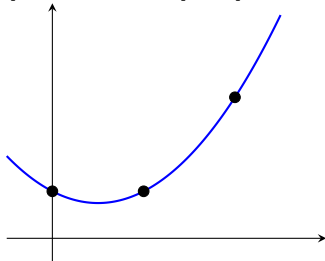
If points $x_0, \dots, x_n$ are distinct, then for arbitrary $y_0, \dots, y_n$, there is a unique polynomial $p(x)$ of degree at most n such that $p(x_i) = y_i$ for $i = 0, \dots, n$.

- Interpolation is a well-defined problem
- If you are given $n + 1$ data points, there can only be one degree n interpolation



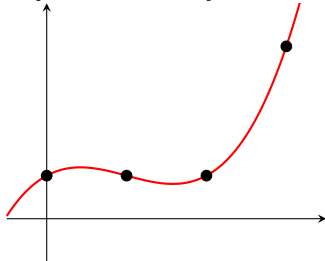
Interpolation is Determined by the Data

3 points $\rightarrow$ unique quadratic



3 data points determine a quadratic.

4 points $\rightarrow$ unique cubic



4 data points determine a cubic.

Key Idea

If your data has $n + 1$ points, then the goal is find a degree n polynomial.



Basis Functions



Basis Functions

Question

So how do you build the polynomials $p(x)$?



Basis Functions

Question

So how do you build the polynomials $p(x)$?



Basis functions = building blocks for $p(x)$



Basis Functions

- A family of functions $\mathcal{F}$ can serve as basis functions for polynomials if the functions are linearly independent.

$$\mathcal{F} = \{f_0(x), f_1(x), f_2(x), f_3(x), \dots, f_n(x)\}$$

- We may take linear combinations of the basis functions in $\mathcal{F}$ to create the polynomial

$$p(x) = a_0f_0(x) + a_1f_1(x) + a_2f_2(x) + a_3f_3(x) + \dots + a_nf_n(x)$$

Key Idea

The basis functions are building blocks for interpolating functions:

$$p(x) = \sum_{i=0}^n \underbrace{a_i}_{\text{weight}} \times \underbrace{f_i(x)}_{\text{basis}}$$

Finding the interpolation boils down to finding the **weights**.

Choice for Basis Functions

Question

So what are the building blocks for polynomials? In other words, who are the members of

$$\mathcal{F} = \{f_0(x), f_1(x), f_2(x), f_3(x), \dots, f_n(x)\}?$$



Choice for Basis Functions

Question

So what are the building blocks for polynomials? In other words, who are the members of

$$\mathcal{F} = \{f_0(x), f_1(x), f_2(x), f_3(x), \dots, f_n(x)\}?$$

- A very natural choice is

$$f_0(x) = 1, \quad f_1(x) = x, \quad f_2(x) = x^2, \quad \dots$$

- This basis can construct any polynomial of the form

$$p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$



Definition: Monomial Basis

Monomials are the family of functions $\mathcal{F} = \{1, x, x^2, x^3, \dots\}$

- Each member of the monomial $f_i(x) \in \mathcal{F}$ has the form

$$f_i(x) = x^i$$

- Monomials naturally form a basis for polynomials of any degree

$$p_n(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$

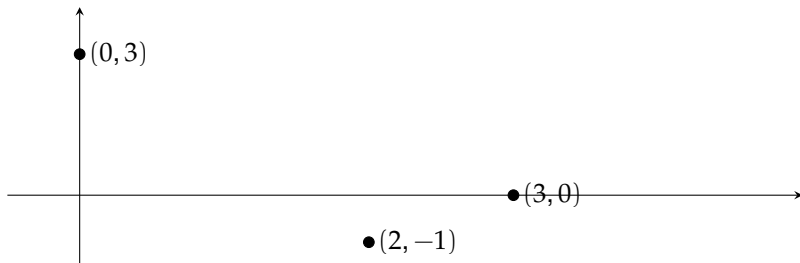
- Next, let's apply interpolation using a monomial basis



Simple Interpolation with 3 Points

Problem

Consider fitting a polynomial through the data points $(0, 3)$, $(2, -1)$ and $(3, 0)$.



- There are $n + 1 = 3$ points
- That means we want to fit a $n = 2$ degree polynomial

$$p(x) = a_0 + a_1x + a_2x^2$$



Simple Interpolation with 3 Points

Problem

Consider fitting a polynomial through the data points $(0, 3)$, $(2, -1)$ and $(3, 0)$.

Goal

Assume the polynomial has the form

$$p(x) = a_0 + a_1x + a_2x^2$$

and find a_0 , a_1 , and a_2 .



Unknowns vs. Conditions

Unknowns

The interpolation is $p(x) = a_0 + a_1x + a_2x^2$, so the unknowns are

$$a_0$$

$$a_1$$

$$a_2$$

Constraints or Conditions

The polynomial must cut through all data points. The conditions are

$$p(0) = 3,$$

$$p(2) = -2,$$

$$p(3) = 0.$$

Well-defined

Since there are 3 unknowns and 3 conditions, this problem is well-defined.

Simple Interpolation with 3 Points

Constraint

The polynomial must satisfy

$$p(0) = 3$$

$$p(2) = -1$$

$$p(3) = 0$$

Substitute $p(x) = a_0 + a_1x + a_2x^2$:

$$a_0 = 3$$

$$a_0 + 2a_1 + 4a_2 = -1$$

$$a_0 + 3a_1 + 9a_2 = 0$$

Observation

Finding the interpolating polynomial reduces to solving a system for the coefficients a_0, a_1, a_2 .

Matrix Form

The interpolation equations can be written as a linear system:

$$\begin{bmatrix} 1 & 0 & 0^2 \\ 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

Interpretation

- Unknowns: polynomial coefficients a_0, a_1, a_2
- Matrix entries depend only on the interpolation nodes x_i
- Right-hand side contains the data y_i

Key Idea

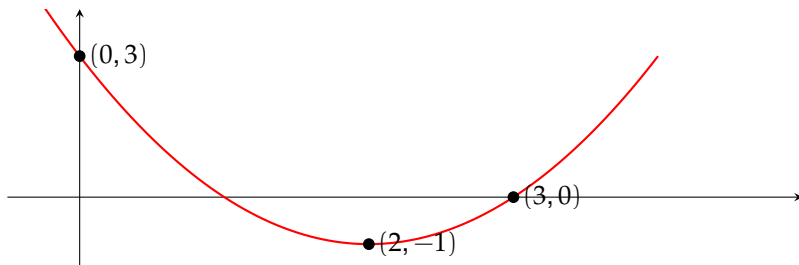
Finding the polynomial interpolation of a set of data (x_i, y_i) is the same as solving $Ax = b$.



Simple Interpolation with 3 Points

Problem

Consider fitting a polynomial through the data points $(0, 3)$, $(2, -1)$ and $(3, 0)$.



- Solving the linear system gives $a_0 = 3$, $a_1 = -4$, $a_2 = 1$ so the fit is

$$p(x) = x^2 - 4x + 3$$



General Interpolation with Monomials

- Our polynomial of degree n has $n + 1$ unknown coefficients

$$p_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

- There are $n + 1$ conditions for the interpolation

$$p_n(x_i) = y_i, \quad \text{for } i = 0, 1, \dots, n$$

- Writing out all equations

$$a_0 + a_1x_0 + a_2x_0^2 + \cdots + a_nx_0^n = y_0$$

$$a_0 + a_1x_1 + a_2x_1^2 + \cdots + a_nx_1^n = y_1$$

$$a_0 + a_1x_2 + a_2x_2^2 + \cdots + a_nx_2^n = y_2$$

$$a_0 + a_1x_3 + a_2x_3^2 + \cdots + a_nx_3^n = y_3$$

$$\vdots$$

$$a_0 + a_1x_n + a_2x_n^2 + \cdots + a_nx_n^n = y_n$$



The Vandermonde

Definition: Vandermonde

The matrix below is called the **Vandermonde matrix**, often labeled V .

- The system in matrix form:

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^n \\ & & & \ddots & \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

- Solving this system gives the coefficients to

$$p_n(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$



Monomial Interpolation

Method: Monomial Interpolation

- **Given:** $n + 1$ data points (x_i, y_i)
 - **Output:** Degree n polynomial cutting through all data points
 - **Idea:** Set up an $(n + 1) \times (n + 1)$ matrix and solve it to find polynomial coefficients
- If you use a **monomial basis**

$$\mathcal{F} = \{1, x, x^2, x^3, \dots\}$$

then your interpolation has the familiar form

$$p_n(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$

- The unknown coefficients $a_0, a_1, \dots, a_n$ can be found by solving the **Vandermonde** system



An issue

Question

Can you identify a possible issue for solving this system?

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^n \\ & & & \ddots & \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

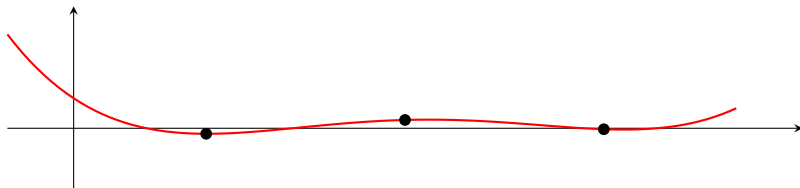
- Suppose the points $x_0, x_1, \dots, x_n$ are between 0 and 1
- What happens if n is large?



Bad Example

Problem

Consider fitting a polynomial through the data points $(\epsilon_1, -0.0045)$, $(\epsilon_2, 0.0068)$, and $(\epsilon_3, -0.0008)$.

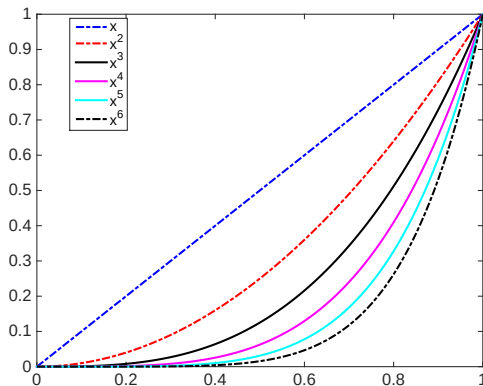


Vandermonde Matrix after underflow

$$\begin{bmatrix} 1 & \epsilon_1 & \epsilon_1^2 \\ 1 & \epsilon_2 & \epsilon_2^2 \\ 1 & \epsilon_3 & \epsilon_3^2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \epsilon_1 & 0 \\ 1 & \epsilon_2 & 0 \\ 1 & \epsilon_3 & 0 \end{bmatrix}$$



III-Conditioning!



Key Idea

- $\text{cond}(V)$ typically grows with n
- If $\text{cond}(V)$ is high, the coefficient $a_0, \dots, a_n$ are computed with poor precision

Another Downside

Problem

Consider fitting a polynomial through the data points $(0, 3)$, $(1, 4)$, $(2, -1)$ and $(3, 0)$.

- It is easy to set up the linear system

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \\ 0 \end{bmatrix}$$

- It is difficult to find the interpolation by hand



Lagrange Interpolation



Another Basis

Monomial Basis

Monomials are the family of functions

$$\mathcal{F} = \{1, x, x^2, x^3, \dots\}$$

- Form a basis for polynomials
- A natural choice
- Suffers from some drawbacks for interpolation

Question

Is there another basis for polynomials?



Consider this function $p(x)$

Question

Suppose you have two data points: (x_0, y_0) and (x_1, y_1) . What is the behavior of the following function?

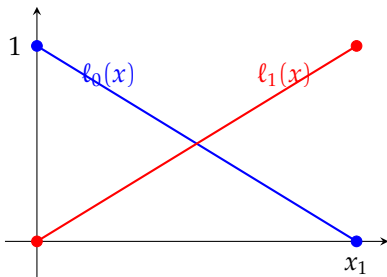
$$p(x) = \left(\frac{x - x_1}{x_0 - x_1} \right) y_0 + \left(\frac{x - x_0}{x_1 - x_0} \right) y_1$$

- $p(x)$ is a degree 1 polynomial
- $\ell_0(x)$ is zero at x_1 and 1 at x_0
- $\ell_1(x)$ is zero at x_0 and 1 at x_1
- Thus $p(x_0) = y_0$ and $p(x_1) = y_1$
- These two are called **Lagrange basis functions**:

$$\ell_0(x) = \frac{x - x_1}{x_0 - x_1} \quad \ell_1(x) = \frac{x - x_0}{x_1 - x_0}$$



Lagrange Basis Functions for Two Points



Key Properties

$$\ell_0(x_0) = 1, \quad \ell_0(x_1) = 0$$

$$\ell_1(x_0) = 0, \quad \ell_1(x_1) = 1$$

- Each basis function is 1 at its own node
- Each basis function is 0 at the other node
- Together they build the interpolating line

$$p(x) = y_0\ell_0(x) + y_1\ell_1(x)$$



Lagrange Basis Functions

Definition: Lagrange Basis Functions

Let $x_0, x_1, \dots, x_n$ be distinct points. **Lagrange basis functions** are a family of functions

$$\mathcal{F} = \{\ell_0(x), \ell_1(x), \dots, \ell_n(x)\}$$

satisfying

$$\ell_k(x) = \prod_{i=0, i \neq k}^n \frac{x - x_i}{x_k - x_i}, \text{ for } k = 0, \dots, n$$

Properties:

- Each member $\ell_k(x)$ is a degree n polynomial
- When you evaluate ℓ_k at x_k , you get 1 $\ell_i(x_i) = 1$
- $\ell_i(x_j) = 0$ whenever $i \neq j$



Lagrange Basis for 3 Points

Example

Find the first Lagrange basis function $\ell_0(x)$ for the data points $(1, 6)$, $(-1, 2)$, and $(2, 12)$.

- First, $\ell_0(x)$ must be zero $x_1 = -1$ and $x_2 = 2$ so we write

$$\ell_0(x) = (x + 1)(x - 2)$$

- Next, we find the correcting scalar. We want $\ell(x) = 1$ at $x_0 = 1$
- Plug in x_0

$$\ell_0(1) = (1 + 1)(1 - 2) = -2$$

- We must *correct* ℓ_0 by scaling it by -2

$$\ell_0(x) = -\frac{1}{2}(x + 1)(x - 2)$$



Sanity Check

Question

Does the function

$$\ell_0(x) = -\frac{1}{2}(x+1)(x-2)$$

satisfy the properties of a Lagrange basis function for

x	1	-1	2	
y	6	2	12	?

- $\ell_0(1) = -\frac{1}{2}(1+1)(1-2) = 1\checkmark$
- $\ell_0(-1) = -\frac{1}{2}(-1+1)(-1-2) = 0\checkmark$
- $\ell_0(2) = -\frac{1}{2}(2+1)(2-2) = 0\checkmark$

Key Idea

- Lagrange basis functions act like a switch: 0 or 1
- $\ell_0(x)$ is switched on at $x = x_0$
- $\ell_0(x)$ is switched off at $x = x_1$ and $x = x_2$

Interpolation

Data

x	1	-1	2
y	6	2	12

- We found $\ell_0(x)$. We can find $\ell_1(x)$ and $\ell_2(x)$ similarly
- Then we can construct the interpolation

$$p(x) = \ell_0(x) + 2\ell_1(x) + 12\ell_2(x)$$

- At $x = x_0$

$$p(x_0) = \ell_0(x_0) + \cancel{2\ell_1(x_0)} + \cancel{12\ell_2(x_0)} = 1$$

- At $x = x_1$

$$p(x_1) = \cancel{\ell_0(x_1)} + 2\ell_1(x_1) + \cancel{12\ell_2(x_1)} = 2$$

- At $x = x_2$

$$p(x_2) = \cancel{\ell_0(x_2)} + \cancel{2\ell_1(x_2)} + 12\ell_2(x_2) = 12$$



Lagrange Interpolation Example

Part 1

Example

Find the Lagrange interpolation for the data set $(1, 6)$, $(-1, 2)$, and $(2, 12)$.

- Calculate ℓ_0 : $k = 0, i = 1, 2$

$$\ell_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x + 1)(x - 2)}{(2)(-1)} = -\frac{1}{2}(x + 1)(x - 2)$$

- Calculate ℓ_1 : $k = 1, i = 0, 2$

$$\ell_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x - 1)(x - 2)}{(-2)(-3)} = \frac{1}{6}(x - 1)(x - 2)$$

- Calculate ℓ_2 : $k = 2, i = 0, 1$

$$\ell_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x - 1)(x + 1)}{(1)(3)} = \frac{1}{3}(x - 1)(x + 1)$$



Lagrange Interpolation Example

Part 2

Example

Write the three Lagrange basis functions for the data points $(1, 6)$, $(-1, 2)$, and $(2, 12)$.

- Then write the linear combination of the Lagrange basis functions

$$\begin{aligned} p(x) &= y_0 \ell_0(x) + y_1 \ell_1(x) + y_2 \ell_2(x) \\ &= \ell_0(x) + 2\ell_1(x) + 12\ell_2(x) \\ &= -\frac{1}{2}(x+1)(x-2) + \frac{2}{6}(x-1)(x-2) + \frac{12}{3}(x-1)(x+1) \end{aligned}$$



Lagrange Interpolation

Key Idea

- Since Lagrange basis functions have a switch-like property,

$$y_k \ell_k(x)$$

will be y_k at $x = x_k$ and be zero at all other points

- We may easily create interpolation polynomials using Lagrange functions as the basis:

$$p(x) = y_0 \ell_0(x) + y_1 \ell_1(x) + \cdots + y_n \ell_n(x)$$

Method: Lagrange Interpolation

- **Given:** Data $n + 1$ points (x_i, y_i) for $i = 0, \dots, n$
- **Output:** Interpolating polynomial $p_n(x)$
- **Idea:** With Lagrange basis functions the interpolation can easily be found by hand. (Solving a linear system is not required.)

Final Remarks



Summary

Monomial Basis

$$\mathcal{F} = \{1, x, x^2, x^3, \dots\}$$

- Leads to a linear system (Vandermonde matrix)
- Can be expensive and numerically unstable

Lagrange Basis

$$\mathcal{F} = \{\ell_0(x), \ell_1(x), \dots, \ell_n(x)\}$$

$$\ell_k(x) = \prod_{i=0, i \neq k}^n \frac{x - x_i}{x_k - x_i}$$

- Avoids solving a linear system
- Gives an explicit formula for $p(x)$
- But has limited flexibility

Limitations of Lagrange Interpolation

- **Adding a new data point is expensive**

- If we add (x_{n+1}, y_{n+1}) , we must recompute all basis functions
- The entire polynomial must be rebuilt from scratch

- **No reuse of previous work**

- We cannot easily update the polynomial incrementally

- **Not ideal for computation**

- Expression becomes complicated for large n
- Difficult to evaluate efficiently

Key Idea

Lagrange interpolation is conceptually simple, but not computationally efficient.



Psuedocode

Just solve a linear system.

```
1 Create vandermonde matrix V
2 Create b vector
3 Solve a=V\b
4 Construct polynomial
```

Listing 1: monomial

Create a domain vector, and construct $p(x)$ as a vector.

```
1 xx = linspace(x0,xn)
2 p = zeros(size(xx))
3 for i=1:n
4     Initialize li
5     for j = 0:n (skip i)
6         li = li + (xx-xi)./(xj-xi)
7     end
8 end
```

Listing 2: monomial



Moving Forward: Newton Interpolation

Idea

Can we build the interpolating polynomial **incrementally**?

- Start with one data point $\{x_0\}$
- Add one point at a time $\{x_0, x_1\}$
- Update the polynomial without recomputing everything
- This leads to a new representation:

$$p(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + \dots$$

- Next lecture, we will introduce a third basis: **the Newton Polynomial basis functions**

