

Lecture 07 - A

§8.1 LU Factorization

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We will answer these questions:

- What are efficient strategies for solving $AX = B$, a multiple right-hand side problem?
- What is the benefit of factorizing $A = LU$ in terms of solving $Ax = b$?
- How is Gaussian elimination expressed in terms of matrix multiplication?
- How do you construct an LU factorization.



Multiple Right Hand Side



Single Solve

Suppose $A \in \mathbb{R}^{n \times n}$. The singular problem

$$Ax = b$$

costs $\mathcal{O}(n^3)$ to solve.

Many applications require solving $Ax = b_i$ repeatedly with the **same matrix** A :

$$Ax^{(1)} = b^{(1)}, \quad Ax^{(2)} = b^{(2)}, \quad \dots, \quad Ax^{(m)} = b^{(m)}.$$



Consider this problem...

Equivalent Problems

- Solving $Ax = b$ for **multiple right hand sides** (RHS), i.e.

$$Ax_1 = b_1, Ax_2 = b_2, \dots, Ax_m = b_m$$

- Solving for X in the equation

$$\underbrace{A}_{n \times n} \underbrace{X}_{n \times m} = \underbrace{B}_{n \times m}.$$

Question

What are the strategies to solve $AX = B$? What will they cost?



Multiple Right Hand Sides

Options

Option A: Do Gaussian elimination (GE) m times

- Pro: Never compute A^{-1}
- Con: But it's expensive $\mathcal{O}(mn^3)$



Multiple Right Hand Sides

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Option B: Compute A^{-1} once and reuse it

- Pro: Get to reuse A^{-1} for each RHS
- Con: but you're computing A^{-1}



Multiple Right Hand Sides

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- Con: but you're computing A^{-1}

Key Idea

- Options A and B are **both bad**
- **Matrix decomposition** combines the best parts of Options A and B



What is Matrix Decomposition?

Definition: Matrix Decomposition

Matrix decomposition or **matrix factorization** is a factorization of a matrix A into a product of matrices, e.g.

$$A = A_1 A_2$$

Three Types

We will learn three types of decompositions

- $A = LU$
- $A = LDL^T$
- $A = LL^T$



LU Solve



LU Factorization

Definition: LU Factorization

LU factorization or **LU decomposition** is a factorization for a $n \times n$ matrix A into a product

$$A = LU$$

where L is lower triangular with ones on the diagonal, and U is upper triangular.

$$L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \ell_{2,1} & 1 & 0 & 0 \\ \ell_{3,1} & \ell_{3,2} & 1 & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ \ell_{n,1} & \ell_{n,2} & \cdots & \ell_{n,n-1} & 1 \end{bmatrix} \quad U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ 0 & u_{2,2} & u_{2,3} & \cdots & u_{2,n} \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & u_{n-1,n} \\ 0 & 0 & & & u_{n,n} \end{bmatrix}$$



Main Advantage

- Usually, solving $Ax = b$ costs $\mathcal{O}(n^3)$, but...

Theorem

If A is factorized as $A = LU$ then $Ax = b$ can be solved in $\mathcal{O}(n^2)$ operations.

- If $A = LU$, then it is an **entire order cheaper** to solve the system

Question

How does this work?



The LU Solve

Steps to solve $Ax = b$:

- 1 Substitute $A = LU$ in $Ax = b$ to get

$$LUx = b$$

- 2 Set $y = Ux$ so that

$$Ly = b$$

- 3 Use forward substitution ($\mathcal{O}(n^2)$) to compute

$$y = L^{-1}b$$

- 4 Use backward substitution ($\mathcal{O}(n^2)$) to compute

$$x = U^{-1}y$$



Revisit Options for Solving $AX = Bs$

Option A: Do Gaussian elimination (GE) m times

- Pro: Never compute A^{-1}
- Con: But it's expensive $\mathcal{O}(mn^3)$

Option B: Compute A^{-1} once and reuse it

- Pro: Get to reuse A^{-1} for each RHS
- Con: but you're computing A^{-1}

Option C: Factorize $A = LU$ once and reuse it

- A^{-1} is never computed
- Cheaper: $\mathcal{O}(n^3 + mn^2)$

Key Idea

LU factorization is safer and faster than both options

Method: LU Solve

- **Given:** Factorization $A = LU$, RHS b
- **Output:** Solution to $Ax = b$
- **Idea:** The LU solve costs only $\mathcal{O}(n^2)$ which is cheaper than GE.

Comments:

- The triangular solves are both cost $\mathcal{O}(n^2)$
- The LU solve costs around $2n^2$ FLOPs
- Especially useful when solving multiple RHS, $m \gg 1$



LU Solve Method

Method: LU Solve

- **Given:** Factorization $A = LU$, RHS b
- **Output:** Solution to $Ax = b$
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Question

So how do you find $A = LU$?



Elimination Matrices



Consider This

Example

Calculate the following matrix vector product

$$Mv = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$$

- Product is

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

- Think of M as an operator that zeros out certain elements in v

$$\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

- M is an example of an **elimination matrix**



Understanding the Operator

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Top Row

Top row $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ says leave the 2 alone.
The 2 remains a 2.

Second Row

Second row $\begin{bmatrix} -2 & 1 & 0 \end{bmatrix}$ says to add -2 copies of 2 to the 4
The 4 becomes $4 - 2 * 2 = 0$.

Third Row

Third row $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$ says to add 1 copy of 2 to the -2.
The -2 becomes $-2 + 1 * 2 = 0$.

Understanding the Operator

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What about a Matrix?

- This matrix is able to eliminate elements under the top element of v

$$Mv = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$



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Question

What is the effect of multiplying M to another matrix?

$$MA = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 2 \\ -2 & -1 & 5 \end{bmatrix}$$



What about a Matrix?

Example

Calculate the following matrix product

$$MA = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 2 \\ -2 & -1 & 5 \end{bmatrix}$$

- First row: do nothing

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 2 \\ -2 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ & & \\ & & \end{bmatrix}$$



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- Second row: add -2 copies of R_1 to R_2

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- Third row: add 1 copy of R_1 to R_3

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 2 \\ -2 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$



Elimination Matrix $\times$ Matrix

Question

What happens when you multiply an elimination matrix to another matrix?

$$MA = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 2 \\ -2 & -1 & 5 \end{bmatrix}$$

- The first row is preserved

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 2 \\ -2 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ & & \\ & & \end{bmatrix}$$



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- Elements under a_{11} are eliminated

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 2 \\ -2 & -1 & 5 \end{bmatrix} = \begin{bmatrix} \boxed{2} & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$



Designing the Elimination Matrix

Example

Find a matrix 4×4 elimination matrix M that eliminates the elements under

$a_{11} = 6$ when multiplied as MA , where $A = \begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix}$

- Formula: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -m_2 & 1 & 0 & 0 \\ -m_3 & 0 & 1 & 0 \\ -m_4 & 0 & 0 & 1 \end{bmatrix}$, where $m_2 = \frac{a_{21}}{a_{11}}$, $m_3 = \frac{a_{31}}{a_{11}}$ and $m_4 = \frac{a_{41}}{a_{11}}$
- Thus our elimination matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boxed{6} & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix} = \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \end{bmatrix}$$



Constructing U



Goal

- We want to construct the U in

$$A = LU$$

- Multiply both sides by L^{-1}

$$L^{-1}A = L^{-1}LU$$

$$L^{-1}A = U$$

Theorem

If L is lower triangular then L^{-1} is also lower triangular.

Key Idea

This equation

$$L^{-1}A = U$$

tells us that we construct U by multiplying A by some *lower triangular matrix*.

What is the lower triangular matrix?

- We know that elimination matrices are lower triangular

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$



What is the lower triangular matrix?

- We know that elimination matrices are lower triangular

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

- But the multiplication doesn't always create an upper triangular

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}}_{M_1} \underbrace{\begin{bmatrix} \boxed{6} & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix}}_A = \underbrace{\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \end{bmatrix}}_{\text{not upper triangular :(}}$$



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- What can we do?



Use Multiple Elimination Matrices

$$\underbrace{\begin{bmatrix} \boxed{x} & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix}}_A \rightarrow \begin{bmatrix} \boxed{x} & x & x & x \\ 0 & x' & x' & x' \\ 0 & x' & x' & x' \\ 0 & x' & x' & x' \end{bmatrix} \text{ multiply by } M_1$$
$$\rightarrow \begin{bmatrix} x & x & x & x \\ 0 & \boxed{x'} & x' & x' \\ 0 & 0 & x'' & x'' \\ 0 & 0 & x'' & x'' \end{bmatrix} \text{ multiply by } M_2$$
$$\rightarrow \begin{bmatrix} x & x & x & x \\ 0 & x' & x' & x' \\ 0 & 0 & \boxed{x''} & x'' \\ 0 & 0 & 0 & x''' \end{bmatrix} \text{ multiply by } M_3$$

Let's call this U

Key Idea

Repeatedly multiply A by **elimination matrices** to make it upper triangular.

How to Construct U

- Repeatedly multiply A from the left by elimination matrices

$$\begin{aligned} & A \\ & M_1 A \\ & M_2 M_1 A \\ & \vdots \\ & (M_{n-1} \dots M_1) A = U \end{aligned}$$

Question

- What does the product $(M_{n-1} \dots M_1)$ look like?
- How do you construct L ?



Some Properties



Formula for Elimination Matrices

- This is the M_1 elimination matrix because it eliminates elements under the first pivot

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

- Notice that this matrix can be written as

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}}_{m^{(1)}e_1^T},$$

where

$$m^{(1)} = \begin{bmatrix} 0 \\ 2 \\ \frac{1}{2} \\ -1 \end{bmatrix}, \quad e_1^T = [1 \quad 0 \quad 0 \quad 0].$$



Elimination Matrix Properties

Elimination Matrix

Elimination matrices have the form

$$M_k = I - m^{(k)} e_k^T,$$

where

- e_k is the k^{th} column of the identity matrix I
- $m^{(k)} = [0, \dots, 0, m_{k+1}, \dots, m_n]'$ vector with multipliers

Property 1: Inverse

The inverse is $M_k^{-1} = I + m^{(k)} e_k^T$

Property 2: Multiplication

Product of two elimination matrices M_k and M_j is a combined elimination matrix

Inverse Property

Theorem

If $M_k = I - m^{(k)}e'_k$ then $M_k^{-1} = I + m^{(k)}e'_k$.

Proof:

$$\begin{aligned}M_k M_k^{-1} &= (I - m^{(k)}e'_k)(I + m^{(k)}e'_k) \\&= II + Im^{(k)}e'_k - m^{(k)}e'_k I - m^{(k)}e'_k m^{(k)}e'_k \\&= I + \underbrace{m^{(k)}e'_k - m^{(k)}e'_k}_{\text{cancel}} - m^{(k)}(e'_k m^{(k)})e'_k \\&= I - m^{(k)} \underbrace{(e'_k m^{(k)})}_{\text{i.p.}=0} e'_k \\&= I\end{aligned}$$

Key Idea

- To find the inverse simply flip the signs of the multipliers.
- For instance:

$$\text{If } M_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \text{then } M_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$



Multiplication Property

Theorem

Suppose M_j and M_k are elementary elimination matrices with $j > k$, then their product is $I - m^{(k)}e_k^T - m^{(j)}e_j^T$.

Proof:

$$\begin{aligned}M_k M_j &= I - m^{(k)}e_k^T - m^{(j)}e_j^T + m^{(k)}e_k^T m^{(j)}e_j^T \\&= I - m^{(k)}e_k^T - m^{(j)}e_j^T + m^{(k)}(e_k^T m^{(j)})e_j^T \\&= I - m^{(k)}e_k^T - m^{(j)}e_j^T\end{aligned}$$

Key Idea

- To multiply $M_k M_j$ simply combine their columns.
- For instance:

$$M_1 M_2 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1/2 & 1 \end{bmatrix}$$



Constructing L and U



Objective

The goal is write A as $A = LU$, where

$$L = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ \ell_{2,1} & 1 & 0 & & 0 \\ \ell_{3,1} & \ell_{3,2} & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ \ell_{n,1} & \ell_{n,2} & \cdots & \ell_{n,n-1} & 1 \end{bmatrix} \quad U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ 0 & u_{2,2} & u_{2,3} & \cdots & u_{2,n} \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & u_{n-1,n} \\ 0 & 0 & & & u_{n,n} \end{bmatrix}$$



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The goal is write A as $A = LU$, where

$$L = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ \ell_{2,1} & 1 & 0 & & 0 \\ \ell_{3,1} & \ell_{3,2} & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ \ell_{n,1} & \ell_{n,2} & \cdots & \ell_{n,n-1} & 1 \end{bmatrix} \quad U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ 0 & u_{2,2} & u_{2,3} & \cdots & u_{2,n} \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & u_{n-1,n} \\ 0 & 0 & & & u_{n,n} \end{bmatrix}$$

Question

Don't we already know a way to make A upper triangular?



LU Factorization

Part 1

1. Repeatedly multiply A from the left by elimination matrices

$$\begin{aligned} & A \\ & M_1 A \\ & M_2 M_1 A \\ & \vdots \\ & \underbrace{(M_{n-1} \dots M_1) A}_{\text{upper triangular}} = U \end{aligned}$$

2. In the end, the product will be upper triangular



LU Factorization

Part 2

3. Multiply both sides from the left by $(M_{n-1} \cdots M_1)^{-1}$:

$$(M_{n-1} \cdots M_1)A = U,$$

$$A = (M_{n-1} \cdots M_1)^{-1}U.$$

4. Reverse the order when taking the inverse:

$$A = M_1^{-1} \cdots M_{n-1}^{-1}U.$$

Inverse of a Product

For square nonsingular matrices P and Q ,

$$(PQ)^{-1} = Q^{-1}P^{-1}.$$



LU Factorization

Part 3

5. Use the two **multiplication and inverse properties**:

$$(M_1^{-1} \dots M_{n-1}^{-1}) = I + \sum_{k=1}^{n-1} m^{(k)} e_k^T = L$$

Invert each elimination matrix by changing the signs of its off-diagonal entries, then combine the columns.

6. We are done with the LU factorization

$$A = (M_1^{-1} \dots M_{n-1}^{-1})U$$

$$A = LU$$

Key Idea

To construct L :

- ▶ Put ones on the diagonal.
- ▶ Store each elimination multiplier $m_{ik} = a_{ik}/a_{kk}$ in position (i, k) .
- ▶ These entries have the opposite signs to the corresponding entries in M_k .

Factorization Example

Part 1

Example

Find the LU factorization of $A = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 4 & -2 \\ 2 & 2 & 13 \end{bmatrix}$.

1. Construct elimination matrix M_1 . Left-multiply it to A .

$$M_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boxed{1} & -2 & 3 \\ -1 & 4 & -2 \\ 2 & 2 & 13 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & 1 \\ 0 & 6 & 7 \end{bmatrix}$$

2. Construct elimination matrix M_2 . Left-multiply it. U is complete.

$$M_2 M_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 0 & \boxed{2} & 1 \\ 0 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} =: U$$



Factorization Example

Part 2

3. Derive a formula for L based on the equation $A = LU$

$$M_2 M_1 A = U$$

$$A = M_1^{-1} M_2^{-1} U$$

$$\therefore L = M_1^{-1} M_2^{-1}$$

4. Construct L using the properties of elimination matrices:

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \implies M_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad \text{just flip sign}$$

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \implies M_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \quad \text{just flip sign}$$

$$L = M_1^{-1} M_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \quad \text{just combine columns}$$



Summary

- The LU factorization of the matrix is

$$\begin{bmatrix} 1 & -2 & 3 \\ -1 & 4 & -2 \\ 2 & 2 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

- You may multiply LU to verify that it equals A
- Main steps:
 - 1 Perform GE with elimination matrices
 - 2 Repeat until the result is upper triangular - that is U
 - 3 Construct L by combining all eliminations but with sign-flipped multipliers



Coded Example

Matlab: `matrix_form_LU.m`

The code demonstrates the LU factorization of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 1 \\ 3 & 4 & 1 \end{bmatrix}$.

- Run one section at a time
- Understand how row operations are coded into elimination matrices

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -5 \\ 0 & 0 & -18 \end{bmatrix}$$



LU versus GE

```
1 given A
2
3 for k = 1...n - 1
4     for i = k + 1...n
5          $xmult = a_{ik}/a_{kk}$ 
6          $a_{ik} = xmult$ 
7         for j = k + 1...n
8              $a_{ij} = a_{ij} - (xmult)a_{kj}$ 
9         end
10        % doesn't affect b
11    end
12 end
```

LU factorization

```
1 given A, b
2
3 for k = 1...n - 1
4     for i = k + 1...n
5          $xmult = a_{ik}/a_{kk}$ 
6          $a_{ik} = 0$ 
7         for j = k + 1...n
8              $a_{ij} = a_{ij} - (xmult)a_{kj}$ 
9         end
10         $b_i = b_i - (xmult)b_k$ 
11    end
12 end
```

GE

They are basically the same algorithm.



LU Factorization

Method: LU Factorization

- **Given:** Square matrix A
- **Output:** Unit lower L and upper triangular U , such that $LU = A$
- **Idea:** Factorization of A comes naturally from GE.

Recap:

- U is naturally constructed as part of Gaussian elimination.

$$U = (M_{n-1} \cdots M_1)A$$

- L is a byproduct of GE adding no extra cost (just save the multipliers)

$$L = M_1^{-1} \cdots M_{n-1}^{-1}$$

Key Idea

LU factorization performs elimination and saves the multipliers: $\mathcal{O}(n^3)$ FLOPs. Once the factors are available, an LU solve uses forward and backward substitution: $\mathcal{O}(n^2)$ FLOPs per right-hand side.

Questions

- 1 Do all matrices have an LU factorization?
- 2 Does LU factorization need pivoting like GE?
- 3 Are there other useful factorizations?

