

Lecture 06 - B

§2.1, 2.2 Pivoting

T. Takahashi

Department of Mathematics and Statistics
University of New Mexico



We will answer these questions:

- How do row and column swaps affect a linear system?
- What is the problem with a small non-zero pivot element?
- What is the difference between partial pivoting, full pivoting, and scaled partial pivoting?



Row and Column Swapping



Naive Gaussian Elimination

Definition: Pivot

The **pivot element** refers to the boxed elements in the matrix. The **pivot row** refers to the row containing pivot element.

- The pivot element controls the **scalar multiplier**
- Algorithmically, the pivot row is the current row in the iteration process
- Naive GE chosen to be the diagonal elements

$$A = \begin{bmatrix} \boxed{0} & 2 & 3 \\ 4 & \boxed{5} & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Naive GE does not work if there is a zero on the diagonal.



Fix: Row Swap

- Recall this example from the previous lecture

$$\left[\begin{array}{ccc|c} \boxed{2} & 4 & -2 & -4 \\ 1 & 2 & 4 & 5 \\ -2 & -3 & 8 & 7 \end{array} \right] \quad \begin{array}{l} R_2 \leftarrow R_2 - \frac{1}{2}R_1 \\ R_3 \leftarrow R_3 + R_1 \end{array}$$

- Fix:** Swap rows $R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & -4 \\ 0 & \boxed{0} & 5 & 7 \\ 0 & 1 & 6 & -3 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 2 & 4 & -2 & -4 \\ 0 & \boxed{1} & 6 & -3 \\ 0 & 0 & 5 & 7 \end{array} \right]$$

- GE is now complete. Do back sub to solve the system.



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- GE is now complete. Do back sub to solve the system.

Question

But why is row swapping allowed?



Row Swapping

Question

What happens when you swap Equations 1 and 2?

- **Problem A:** Solve

$$\begin{array}{lcl} \text{Equation 1:} & x + y = 3 & \\ \text{Equation 2:} & -x + 2y = 0 & \end{array} \quad \text{or} \quad \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$



Row Swapping

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- **Problem B:** Solve

$$\begin{array}{lcl} \text{Equation 2:} & -x + 2y = 0 & \\ \text{Equation 1:} & x + y = 3 & \end{array} \quad \text{or} \quad \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Key Idea

Swapping two complete equations, including their right-hand sides, preserves the solution set.

What about Column Swapping?

- Suppose your system is

$$\begin{array}{lcl} \text{Equation 1:} & x + y = 3 & \\ \text{Equation 2:} & -x + 2y = 0 & \end{array} \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & -3 \\ -1 & 2 & 0 \end{array} \right]$$



What about Column Swapping?

- Suppose your system is

$$\begin{array}{lcl} \text{Equation 1:} & x + y = 3 & \\ \text{Equation 2:} & -x + 2y = 0 & \end{array} \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & -3 \\ -1 & 2 & 0 \end{array} \right]$$

- Are you allowed to swap columns?

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ -1 & 2 & 0 \end{array} \right] \xrightarrow{C_1 \leftrightarrow C_2} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & -1 & 0 \end{array} \right].$$



What about Column Swapping?

- **Problem A:**

$$\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{array}{ll} \text{Equation 1:} & x + y = 3 \\ \text{Equation 2:} & -x + 2y = 0 \end{array}$$

Solution: $x = 2, y = 1$

- **Problem B:** Row-swapped version of Problem A

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{array}{ll} \text{Equation 1:} & x + y = 3 \\ \text{Equation 2:} & 2x - y = 0 \end{array}$$

Solution: $y = 2, x = 1$

Key Idea

Column swapping is *allowed*, but you must track the *permutation*.



Swapping

- Row swapping is free

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & -4 \\ 1 & 2 & 4 & 5 \\ -2 & -3 & 8 & 7 \end{array} \right] \quad R_2 \leftrightarrow R_3 \quad \left[\begin{array}{ccc|c} 2 & 4 & -2 & -4 \\ -2 & -3 & 8 & 7 \\ 1 & 2 & 4 & 5 \end{array} \right]$$

- Column swapping requires you to memorize permutations

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & -4 \\ 1 & 2 & 4 & 5 \\ -2 & -3 & 8 & 7 \end{array} \right] \quad \begin{array}{l} C_2 \leftrightarrow C_3 \\ x_2 \leftrightarrow x_3 \end{array} \quad \left[\begin{array}{ccc|c} 2 & -2 & 4 & -4 \\ 1 & 4 & 2 & 5 \\ -2 & 8 & -3 & 7 \end{array} \right]$$



Small Pivots



Review: Adding Numbers of Different Sizes

- A sufficiently small contribution can disappear when a floating-point sum is rounded.
- In double precision, choose

$$\epsilon = 2^{-56}, \quad M = \frac{1}{\epsilon} = 2^{56}.$$

Then

$$\text{fl}(1 + \epsilon) = 1, \quad \text{fl}(M + 1) = M.$$

```
1 e = 2^(-56);  
2 M = 1/e;  
3  
4 1 + e == 1  
5 M + 1 == M
```

Key Idea

Floating-point spacing determines whether the smaller contribution survives rounding.

System with a Small Parameter

Problem

$$\begin{aligned}y &= 2, \\ x + 2y &= 6.\end{aligned}$$

- Notice that there is one solution $x = y = 2$
- Now consider the the slightly perturbed problem for

$$0 < \epsilon \ll 1$$

Perturbed Problem

$$\begin{aligned}\epsilon x + y &= 2 \\ x + 2y &= 6,\end{aligned} \quad 0 < \epsilon \ll 1$$

- The solution should still be around $x \approx y \approx 2$



Perturbed Problem

$$\epsilon x + y = 2$$

$$x + 2y = 6$$

- Suppose we solve this with Naive GE

$$\left[\begin{array}{cc|c} \boxed{\epsilon} & 1 & 2 \\ 1 & 2 & 6 \end{array} \right] \quad R_2 \leftarrow -\frac{1}{\epsilon}R_1 + R_2$$

- In *exact* math, this results in

$$\left[\begin{array}{cc|c} \epsilon & 1 & 2 \\ 0 & 2 - \frac{1}{\epsilon} & 6 - \frac{2}{\epsilon} \end{array} \right]$$



GE with Small Pivots

Perturbed Problem

$$\epsilon x + y = 2$$

$$x + 2y = 6$$

- Suppose we solve this with Naive GE

$$\left[\begin{array}{cc|c} \epsilon & 1 & 2 \\ 1 & 2 & 6 \end{array} \right] \quad R_2 \leftarrow -\frac{1}{\epsilon}R_1 + R_2$$

- In *exact* math, this results in

$$\left[\begin{array}{cc|c} \epsilon & 1 & 2 \\ 0 & 2 - \frac{1}{\epsilon} & 6 - \frac{2}{\epsilon} \end{array} \right]$$

Question

How would this look in Matlab?

GE with Small Pivots

- If we used floating point operations $\text{fl}()$

$$\left[\begin{array}{cc|c} \epsilon & 1 & 2 \\ 0 & \text{fl}(2 - \frac{1}{\epsilon}) & \text{fl}(6 - \frac{2}{\epsilon}) \end{array} \right] \rightarrow \left[\begin{array}{cc|c} \epsilon & 1 & 2 \\ 0 & -\frac{1}{\epsilon} & -\frac{2}{\epsilon} \end{array} \right]$$

- Next, let $M = 1/\epsilon$

$$\left[\begin{array}{cc|c} \epsilon & 1 & 2 \\ 0 & -M & -2M \end{array} \right] \rightarrow \begin{cases} \epsilon x + y = 2 \\ -My = -2M \end{cases}$$

- That means $y = 2M/M = 2$ (which is what we expect)



GE with Small Pivots

- If we used floating point operations $\text{fl}()$

$$\left[\begin{array}{cc|c} \epsilon & 1 & 2 \\ 0 & \text{fl}(2 - \frac{1}{\epsilon}) & \text{fl}(6 - \frac{2}{\epsilon}) \end{array} \right] \rightarrow \left[\begin{array}{cc|c} \epsilon & 1 & 2 \\ 0 & -\frac{1}{\epsilon} & -\frac{2}{\epsilon} \end{array} \right]$$

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- That means $y = 2M/M = 2$ (which is what we expect)

Problem!

Naive elimination in double precision gives $y = 2$ and $x = 0$, although the exact solution has $x \approx y \approx 2$.

Why Small Pivots Can Be Dangerous

- Small pivot $\implies$ large multipliers

$$\left[\begin{array}{c|ccc} \epsilon & 1 & 1 & 1 \\ 1 & a & b & c \\ 1 & d & e & f \end{array} \right], \quad M = \frac{1}{\epsilon}.$$

- Elimination subtracts large quantities:

$$\left[\begin{array}{c|ccc} \epsilon & 1 & 1 & 1 \\ 0 & a - M & b - M & c - M \\ 0 & d - M & e - M & f - M \end{array} \right].$$

- If the entries are too small then

$$\xrightarrow{\text{floating-point rounding}} \left[\begin{array}{c|ccc} \epsilon & 1 & 1 & 1 \\ 0 & -M & -M & -M \\ 0 & -M & -M & -M \end{array} \right].$$

Key Idea

Large elimination multipliers can cause partial or complete loss of smaller contributions through rounding.

Choosing Better Pivots

- A zero pivot prevents division during elimination.
- A pivot that is small relative to other entries can produce large multipliers and a loss of accuracy.
- Row and column swaps let us change which entry becomes the pivot.

Objective

Develop a **systematic rule** for choosing pivots that avoids zeros whenever possible and keeps elimination multipliers from becoming too large.

Question

How should we choose the pivot at each elimination step?



Partial Pivoting



Pivoting Strategies

Goal

Learn how to algorithmically avoid zero and small pivots.

Types

- 1 Partial pivoting
- 2 Full pivoting
- 3 Scaled partial pivoting

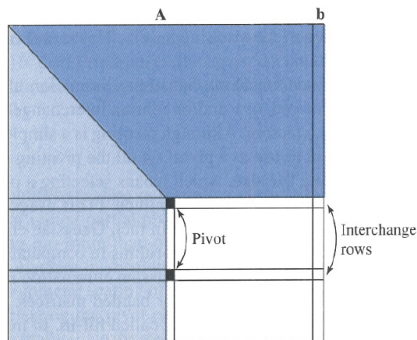
Definition: Partial Pivoting

At elimination step i , **partial pivoting** selects the entry with the largest absolute value in column i , among rows $i, \dots, n$. Its row is swapped with row i .

- Exchanging rows does not affect the order of the x_i
- Increased stability over naive GE
- Easy to implement



Partial Pivoting



At elimination step i :

- 1 Find a row $k \in \{i, \dots, n\}$ such that

$$|a_{ki}| = \max_{j=i, \dots, n} |a_{ji}|.$$

- 2 Swap rows R_i and R_k , including their right-hand-side entries.
- 3 Proceed with elimination below the pivot.



Why Choose the Largest Pivot?

- Elimination below the pivot uses

$$R_j \leftarrow R_j - m_{ji}R_i, \quad m_{ji} = \frac{a_{ji}}{a_{ii}}.$$

- After partial pivoting, the pivot satisfies

$$|a_{ii}| \geq |a_{ji}|, \quad j > i.$$

Therefore, assuming the pivot is nonzero,

$$|m_{ji}| \leq 1.$$

Key Idea

Big pivot $\implies$ small multiplier



Example

Part 1

Example

Perform GE with partial pivoting on the system $Ax = b$ where

$$A = \begin{bmatrix} 0 & 2 & 4 \\ 2 & -3 & 4 \\ -4 & 8 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix}.$$

- Compare the pivot element to all below it in absolute value
 $|-4| > |2| > 0 \implies R_1 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} \boxed{0} & 2 & 4 & 2 \\ 2 & -3 & 4 & -2 \\ -4 & 8 & 2 & 6 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} \boxed{-4} & 8 & 2 & 6 \\ 2 & -3 & 4 & -2 \\ 0 & 2 & 4 & 2 \end{array} \right].$$

- Proceed with elimination $R_2 \leftarrow R_2 + \frac{1}{2}R_1$

$$\left[\begin{array}{ccc|c} \boxed{-4} & 8 & 2 & 6 \\ 2 & -3 & 4 & -2 \\ 0 & 2 & 4 & 2 \end{array} \right] \xrightarrow{R_2 \leftarrow R_2 + \frac{1}{2}R_1} \left[\begin{array}{ccc|c} \boxed{-4} & 8 & 2 & 6 \\ 0 & 1 & 5 & 1 \\ 0 & 2 & 4 & 2 \end{array} \right].$$



Example

Part 2

- In column 2, compare $|1|$ and $|2|$. Since $|2| > |1|$, swap $R_2 \leftrightarrow R_3$:

$$\left[\begin{array}{ccc|c} -4 & 8 & 2 & 6 \\ 0 & \boxed{1} & 5 & 1 \\ 0 & 2 & 4 & 2 \end{array} \right] \longrightarrow \left[\begin{array}{ccc|c} -4 & 8 & 2 & 6 \\ 0 & \boxed{2} & 4 & 2 \\ 0 & 1 & 5 & 1 \end{array} \right].$$

- Eliminate using $R_3 \leftarrow R_3 - \frac{1}{2}R_2$:

$$\left[\begin{array}{ccc|c} -4 & 8 & 2 & 6 \\ 0 & \boxed{2} & 4 & 2 \\ 0 & 1 & 5 & 1 \end{array} \right] \longrightarrow \left[\begin{array}{ccc|c} -4 & 8 & 2 & 6 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & \boxed{3} & 0 \end{array} \right].$$

- Elimination is complete. Backward substitution gives

$$x_3 = 0, \quad x_2 = 1, \quad x_1 = \frac{1}{2}.$$



A Limitation of Partial Pivoting

Example

Perform GE with partial pivoting on the system using floating point arithmetic assuming $M \gg 1$

$$\left[\begin{array}{cc|c} 2 & 2M & 2M \\ 1 & 1 & 2 \end{array} \right]$$

- No row swap. Do elimination $R_2 \leftarrow R_2 - \frac{1}{2}R_1$

$$\left[\begin{array}{cc|c} \boxed{2} & 2M & 2M \\ 1 & 1 & 2 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 2 & 2M & 2M \\ 0 & 1-M & 2-M \end{array} \right]$$

- Perform floating point arithmetic

$$\left[\begin{array}{cc|c} 2 & 2M & 2M \\ 0 & \text{fl}(1-M) & \text{fl}(2-M) \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 2 & 2M & 2M \\ 0 & -M & -M \end{array} \right]$$

so the solution would now be $y = 1$ and $x = 0$.

- The true answer without the rounding is around $x \approx y \approx 1$



Takeaway: What Partial Pivoting Misses

- Partial pivoting examines only the current pivot column.
- Partial pivoting updates numbers as follows

$$a_{jk} \leftarrow a_{jk} - \underbrace{m_{ji}}_{\text{small}} \times \underbrace{a_{ik}}_{\text{large}}.$$

Even when $|m_{ji}|$ is small, a large a_{ik} can still cause large errors!

- All entries below a_{ik} receive loss in precision

Key Idea

Partial pivoting controls the multipliers, but does not check the size of other entries in the pivot row.



Full Pivoting



Definition: Full (or Complete) Pivoting

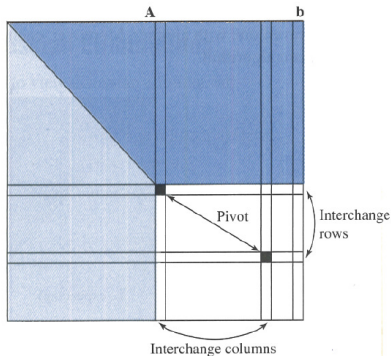
At elimination step i , **full pivoting** chooses the entry with the largest absolute value in the remaining submatrix

$$A_{i:n,i:n}.$$

- Search includes the current pivot row and column.
- Move the chosen entry to position (i, i) using row and column swaps as needed.
- Row swaps also reorder the right-hand side.
- Column swaps require tracking the order of the unknowns.
- Searching the entire remaining submatrix requires more comparisons than partial pivoting.



Full Pivoting Graphic



At elimination step i :

- 1 Find indices $p, q \in \{i, \dots, n\}$ such that

$$|a_{pq}| = \max_{j,k=i,\dots,n} |a_{jk}|.$$

- 2 Swap rows R_i and R_p , swap columns C_i and C_q
- 3 Proceed with elimination below the pivot.



Full Pivoting

Example

Example

Choose the first pivot for

$$A = \begin{bmatrix} 0 & 2 & 4 \\ 2 & 5 & 6 \\ -4 & 8 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}.$$

How must we track the unknowns?

- The largest absolute entry is 8, at position (3, 2). Swap

$$R_1 \leftrightarrow R_3, \quad C_1 \leftrightarrow C_2.$$

- The resulting system is

$$\begin{bmatrix} 8 & -4 & 2 \\ 5 & 2 & 6 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}.$$

- Column swap means solution swap $x_2 \leftrightarrow x_1$.



Cost of Pivoting



Cost Analysis

Finding the Largest Element

```
1 given vector  $v$ , length  $m$ 
2
3 largest =  $|v_1|$ 
4 for  $k = 2, \dots, m$ 
5     if ( $|v_k| > \text{largest}$ ) then
6         largest =  $|v_k|$ 
7     end
8 end
```

Listing 1: max function

- Here we count comparisons between magnitudes, not arithmetic FLOPs.
- Cost to find largest element in vector of size m : $m - 1$ comparisons
- Cost = $\mathcal{O}(m)$



Finding the Largest Element

Suppose x is a vector of size m .

- It costs exactly $m - 1$ floating point comparisons
- The cost = $\mathcal{O}(m)$

Suppose P is a matrix of size $m \times m$.

- It costs exactly $m^2 - 1$ comparisons
- The cost = $\mathcal{O}(m^2)$

Question

What is the cost of full pivoting versus partial pivoting?

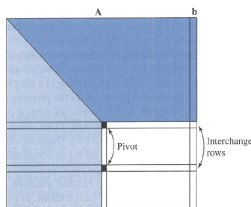


Cost of Partial Pivoting

Suppose the full matrix is $n \times n$

- For the first pivot, compare against $n - 1$ elements
- Second time, compare $n - 2$ elements
- Third time, $n - 3$ elements...
- Total number of pivot-search comparisons:

$$C_{\text{partial}} = (n - 1) + (n - 2) + \cdots + 2 + 1 = \sum_{j=1}^{n-1} j = \frac{n(n-1)}{2} = \mathcal{O}(n^2).$$



Cost of Full Pivoting

- First pivot: search an $n \times n$ submatrix, requiring $n^2 - 1$ comparisons.
- Second pivot: search an $(n - 1) \times (n - 1)$ submatrix.
- Continue down to a 2×2 submatrix. The final 1×1 requires no comparisons.
- Total number of pivot-search comparisons:

$$\begin{aligned}C_{\text{full}} &= \sum_{j=2}^n (j^2 - 1) \\&= \frac{n(n+1)(2n+1)}{6} - n \\&= \frac{1}{3}n^3 + \mathcal{O}(n^2).\end{aligned}$$



Cost Analysis

For dense $n \times n$ systems:

- Gaussian elimination requires

$$\frac{2}{3}n^3 + \mathcal{O}(n^2) \quad \text{arithmetic FLOPs.}$$

- Partial pivoting adds

$$\frac{1}{2}n^2 + \mathcal{O}(n) \quad \text{pivot-search comparisons.}$$

- Full pivoting adds

$$\frac{1}{3}n^3 + \mathcal{O}(n^2) \quad \text{pivot-search comparisons.}$$

Key Idea

Partial pivoting adds a lower-order search cost. Full pivoting adds a search cost of the same order as elimination itself.

Cost Comparison

Consider the following

Suppose it took 60,000 seconds to solve $Ax = b$ with **Naive GE**.

- If Naive GE costs 60,000 then

$$\frac{2}{3}n^3 = 60000$$

$$n^3 = \frac{3}{2}60000 = 90000$$

$$n^2 = (90000)^{2/3} \approx 2000$$

- **Partial pivoting**: add 2,000 extra seconds (30 minutes)
- **Full pivoting**: add extra 30,000 seconds (8 hours)



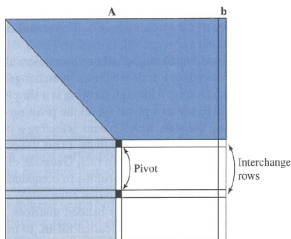
Scaled Partial Pivoting



A compromise between partial and full

Definition: Scaled Partial Pivoting

Scaled partial pivoting is a strategy where pivot element is chosen to be the largest number (in abs. value) within the column below the pivot scaled by the largest element in the row



- Calculate $s_i = \text{the largest entry in row } i \text{ in absolute value}$
- Swap pivot row with largest **scaled pivot value** $\max_j \left| \frac{a_{ij}}{s_i} \right|$

Example

Perform GE with scaled partial pivoting to triangularize the following system

$$2x_1 + 6x_2 - 2x_3 = -4$$

$$x_1 + 3x_2 - 10x_3 = 0$$

$$-2x_1 - 4x_2 + 4x_3 = 8$$

- Start by computing the largest value in each row (not including b_i)

$$\left[\begin{array}{ccc|c} 2 & 6 & -2 & -4 \\ 1 & 3 & -10 & 0 \\ -2 & -4 & 4 & 8 \end{array} \right] \quad \begin{array}{l} R_1 : 2, 6, 2 \\ R_2 : 1, 3, 10 \\ R_3 : 2, 4, 4 \end{array}$$

- These values are calculated *only once*

$$s_1 = 6, \quad s_2 = 10, \quad s_3 = 4$$



SPP By Hand

Part 2

- Compare the scaled pivot values at and below $\boxed{2}$

$$\left[\begin{array}{ccc|c} \boxed{2} & 6 & -2 & -4 \\ 1 & 3 & -10 & 0 \\ -2 & -4 & 4 & 8 \end{array} \right] \quad \frac{|a_{11}|}{s_1} = \frac{2}{6}, \quad \frac{|a_{21}|}{s_2} = \frac{1}{10}, \quad \frac{|a_{31}|}{s_3} = \frac{2}{4}$$

- Largest ratio is $\frac{2}{4}$, so swap $R_1 \leftrightarrow R_3$.

$$\left[\begin{array}{ccc|c} \boxed{2} & 6 & -2 & -4 \\ 1 & 3 & -10 & 0 \\ -2 & -4 & 4 & 8 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} \boxed{-2} & -4 & 4 & 8 \\ 1 & 3 & -10 & 0 \\ 2 & 6 & -2 & -4 \end{array} \right]$$

- Because $R_1 \leftrightarrow R_3$, we must also swap $s_1 \leftrightarrow s_3$

$$\begin{aligned} \text{old :} s_1 &= 6, & s_2 &= 10, & s_3 &= 4 \\ \text{new :} s_1 &= 4, & s_2 &= 10, & s_3 &= 6 \end{aligned}$$



SPP By Hand

Part 3

- Do elimination $R_2 \leftarrow R_2 + \frac{1}{2}R_1$ and $R_3 \leftarrow R_3 + R_1$

$$\left[\begin{array}{ccc|c} \boxed{-2} & -4 & 4 & 8 \\ 1 & 3 & -10 & 0 \\ 2 & 6 & -2 & -4 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} \boxed{-2} & -4 & 4 & 8 \\ 0 & 1 & -8 & 4 \\ 0 & 2 & 2 & 4 \end{array} \right]$$

- Compare the scaled pivot values at and below $\boxed{1}$

$$\left[\begin{array}{ccc|c} -2 & -4 & 4 & 8 \\ 0 & \boxed{1} & -8 & 4 \\ 0 & 2 & 2 & 4 \end{array} \right] \quad \frac{|a_{22}|}{s_2} = \frac{1}{10}, \quad \frac{|a_{32}|}{s_3} = \frac{2}{6}$$

- The largest ratio is $\frac{2}{6}$, so swap $R_2 \leftrightarrow R_3$.

$$\left[\begin{array}{ccc|c} -2 & -4 & 4 & 8 \\ 0 & \boxed{1} & -8 & 4 \\ 0 & 2 & 2 & 4 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} -2 & -4 & 4 & 8 \\ 0 & \boxed{2} & 2 & 4 \\ 0 & 1 & -8 & 4 \end{array} \right]$$

- Carry on with final elimination. Done.



Cost of Scaled Partial Pivoting

- Compute the row scales once:

$$n(n-1) = \mathcal{O}(n^2) \quad \text{comparisons.}$$

- Evaluate the candidate ratios at each step:

$$\sum_{m=2}^n m = \frac{n(n+1)}{2} - 1 = \mathcal{O}(n^2) \quad \text{divisions.}$$

- Select the largest ratio at each step:

$$\sum_{m=2}^n (m-1) = \frac{n(n-1)}{2} = \mathcal{O}(n^2) \quad \text{comparisons.}$$

Key Idea

Scaled partial pivoting adds $\mathcal{O}(n^2)$ comparisons because the row scales are computed only **once**.

Comparing the Pivoting Techniques

Question

Can you list the advantages and disadvantages of PP, FP, and SPP with respect to each other?

Factors to consider:

- Pivot-search cost: comparisons and arithmetic
- Protection against rounding errors
- Additional storage and permutation bookkeeping
- Implementation effort



The truth about pivoting



Key Idea

In practice, most gravitate to partial pivoting because it is stable enough without the implementation overhead