

Lecture 05 - A

Linear Algebra Fundamentals

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Goals

We will answer these questions:

- How do you multiply a matrix and a vector?
- What is the difference between the row and column views of the multiplication Ax ?
- What is a linear system?
- What does it mean for a matrix to be invertible?
- What is the fundamental theorem of linear algebra?



Notation



Matrix Notation

- A **matrix** $A \in \mathbb{R}^{m \times n}$ has the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix}$$

- The entry in row i and column j is denoted

$$a_{ij}.$$

- May also write $(A)_{ij}$



Vector Notation

- A **vector** $v \in \mathbb{R}^n$ with n components has the form

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- The i^{th} element in v is denoted

$$v_i$$

- Vectors are always assumed to be *columns* ($n \times 1$)
- We can **transpose** a vector to make it into a *row*

$$v' = v^T = [v_1 \quad v_2 \quad \cdots \quad v_n]$$



Rows and Columns

As a collection of column vectors

$$A = \left[\begin{array}{c|c|c|c} a_{(1)} & a_{(2)} & \cdots & a_{(n)} \end{array} \right]$$

As a collection of row vectors

$$A = \left[\begin{array}{c} a'_{(1)} \\ \hline a'_{(2)} \\ \hline \vdots \\ \hline a'_{(m)} \end{array} \right]$$

- We may view matrices as a collection of columns or rows



Basic Operations

Vector Addition

Add two vectors $w = u + v$

$$w_i = u_i + v_i$$

Scalar multiplication

Multiply a vector v by a scalar $\alpha \in \mathbb{R}$ $w = \alpha v$

$$w_i = \alpha v_i$$

Definition: Linear Combination

A **linear combination** of vectors $u_1, u_2, \dots, u_k \in \mathbb{R}^n$ is an expression

$$\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_k u_k = \sum_{j=1}^k \alpha_j u_j,$$

where $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ are scalars.

Norms



Vector Norm

- For a scalar $x \in \mathbb{R}$, the absolute value $|x|$ measures its size.
- For a vector $v \in \mathbb{R}^n$, we use a **norm** to measure its size.

Definition: Euclidean Norm

The **Euclidean norm** (or 2-norm) of

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

is

$$\|v\|_2 = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Key Idea

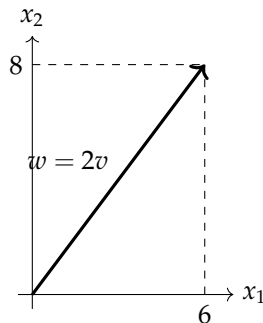
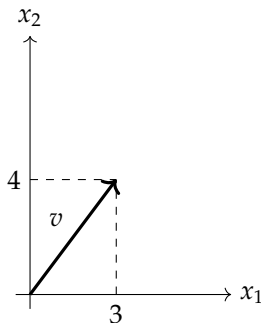
The norm $\|v\|_2$ is the length of the vector v .

Vector Norm

Geometric Interpretation

- Consider the vectors

$$v = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \text{and} \quad w = 2v = \begin{bmatrix} 6 \\ 8 \end{bmatrix}.$$



- Notice their norms:

$$\|v\|_2 = \sqrt{3^2 + 4^2} = 5, \quad \|w\|_2 = \sqrt{6^2 + 8^2} = 10.$$



Scaling a Vector

- Suppose we multiply a vector by a scalar:

$$w = \alpha v.$$

- What happens to its length?

$$\begin{aligned}\|\alpha v\|_2 &= \sqrt{(\alpha v_1)^2 + \cdots + (\alpha v_n)^2} \\ &= |\alpha| \sqrt{v_1^2 + \cdots + v_n^2} \\ &= \boxed{|\alpha| \|v\|_2}.\end{aligned}$$

- Length is scaled by α



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Question

How can we scale a vector to have length 1 without changing its direction?

Definition: Normalized Vector

For a nonzero vector v , its **normalized vector** is

$$\hat{v} = \frac{1}{\|v\|_2} v.$$



Definition: Normalized Vector

For a nonzero vector v , its **normalized vector** is

$$\hat{v} = \frac{1}{\|v\|_2} v.$$

Its length is

$$\|\hat{v}\|_2 = \left\| \frac{1}{\|v\|_2} v \right\|_2 = \frac{1}{\|v\|_2} \|v\|_2 = 1.$$

Key Idea

Normalization changes the length of a vector to 1 while preserving its direction.



Normalization

Example

Example

Normalize the vector

$$v = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

First, compute its norm:

$$\|v\|_2 = \sqrt{3^2 + 4^2} = 5.$$

Therefore,

$$\hat{v} = \frac{v}{\|v\|_2} = \frac{1}{5} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}.$$

Check:

$$\|\hat{v}\|_2 = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = 1.$$



Linear Algebra Multiplication



Product of Vectors

Inner Product

The **inner product** (row $\times$ column) results in a scalar

$$u'v = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \sum_{i=1}^n u_i v_i$$

Outer Product

For $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$, the **outer product** (column $\times$ row) is an $m \times n$ matrix:

$$uv' = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} u_1 v_1 & u_1 v_2 & \cdots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & \cdots & u_2 v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_m v_1 & u_m v_2 & \cdots & u_m v_n \end{bmatrix}.$$

Matrix–Vector Product

- Multiply a matrix $A \in \mathbb{R}^{m \times n}$ by a vector $x \in \mathbb{R}^n$

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- Inner dimensions must agree

$$\begin{array}{ccc} A & x & = & b \\ [m \times n] & [n \times 1] & = & [m \times 1] \end{array}$$



Two Ways of Viewing Matrix-Vector Product

- The Row View treats b as a vector of inner products

$$\left[\begin{array}{c} a'_{(1)} \\ \hline \vdots \\ \hline a'_{(m)} \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

- The Column View treats b as a linear combination of the columns of A

$$\left[\begin{array}{c|c|c|c} a_{(1)} & a_{(2)} & \cdots & a_{(n)} \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$$



In the Row View

- Treat A as a collection of rows

$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix} \begin{bmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{bmatrix} = \begin{bmatrix} \boxed{\bullet} \end{bmatrix}$$

$a'_{(i)} \quad x \quad y_i$

- Formula for the product is

$$y_i = a'_{(i)}x = \sum_{j=1}^n a_{ij}x_j$$



Row View Example

Example

Calculate the product Ax in the row view

$$\begin{bmatrix} 5 & 1 & 0 \\ 0 & 4 & -2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$$

Take the inner product of each row with the vector

$$= \begin{bmatrix} (5)(2) & + & (1)(0) & + & (0)(-3) \\ (0)(2) & + & (4)(0) & + & (-2)(-3) \\ (1)(2) & + & (2)(0) & + & (3)(-3) \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \\ -7 \end{bmatrix}$$



In the Column View

- Treat A as a collection of columns

$$\left[\begin{array}{c|c|c|c} a_{(1)} & a_{(2)} & \cdots & a_{(n)} \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$$

- The product is a linear combination of the columns

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$$



Column View Example

Example

Calculate the product Ax using the column view

$$\begin{bmatrix} 5 & 1 & 0 \\ 0 & 4 & -2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$$

Add up the columns of A multiplied by the scalar from the vector x

$$\begin{aligned} Ax &= 2 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 10 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 6 \\ -9 \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \\ -7 \end{bmatrix} \end{aligned}$$

Same answer as the row view calculation



Column View

- In the column view, Ax is a linear combination of the columns of A :

$$Ax = \sum_{j=1}^n x_j a_{(j)}.$$

Definition: Range, Column Space, and Rank

1. The **range** or **column space** of $A \in \mathbb{R}^{m \times n}$ is

$$\text{range}(A) = \{Ax : x \in \mathbb{R}^n\} = \text{span}\{a_{(1)}, \dots, a_{(n)}\}.$$

2. The **span** of vectors $v_1, \dots, v_k$ is the set of all their linear combinations:

$$\text{span}\{v_1, \dots, v_k\} = \left\{ \sum_{j=1}^k \alpha_j v_j : \alpha_1, \dots, \alpha_k \in \mathbb{R} \right\}.$$

3. Its dimension is the **rank** of A .

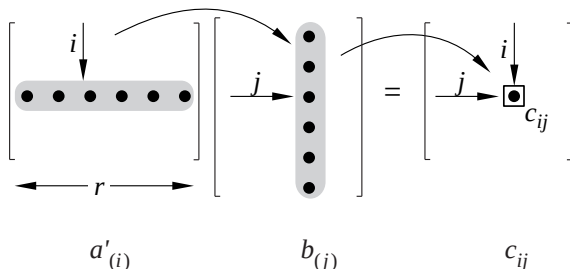
Matrix-Matrix Multiplication

- For $A \in \mathbb{R}^{m \times r}$ and $B \in \mathbb{R}^{r \times n}$, the product $C = AB$ has entries

$$c_{ij} = \sum_{k=1}^r a_{ik} b_{kj}.$$

Each entry is the inner product of row i of A and column j of B .

- Graphically,



- The inner product view of the matrix–matrix product is easier to use for hand calculations.



Compatibility Requirement

- **Compatibility Requirement** - Inner dimensions of A and B must agree

$$\begin{array}{ccc} A & B & = & C \\ [m \times r] & [r \times n] & = & [m \times n] \end{array}$$

- In words,

"The width of A must match the height of B "

- The dimensions of C are given by the outer dimensions.
- Reversing the order may produce an undefined product or a matrix of a different size. Even for square matrices of the same size, in general,

$$AB \neq BA.$$



Matrix and Vector Product Summary

Inner product

$$\begin{bmatrix} \bullet & \bullet & \bullet \end{bmatrix} \begin{bmatrix} \bullet \\ \bullet \\ \bullet \end{bmatrix} = \begin{bmatrix} \bullet \end{bmatrix}$$

Matrix–Vector product

$$\begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} \begin{bmatrix} \bullet \\ \bullet \\ \bullet \end{bmatrix} = \begin{bmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \end{bmatrix}$$

Vector–Matrix product

$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet \end{bmatrix} \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet \end{bmatrix}$$



Matrix and Vector Product Summary

Outer product

$$\begin{bmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \end{bmatrix} \begin{bmatrix} \bullet & \bullet & \bullet \end{bmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix}$$

Matrix-Matrix product

$$\begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix}$$



Linear Dependence



Dependent Vectors

Definition: Dependent Vectors

Two vectors u and v are **linearly dependent** if at least one is a scalar multiple of the other:

$$u = cv \quad \text{or} \quad v = cu \quad \text{for some } c \in \mathbb{R}.$$

- Example:

$$u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

- We see that $u = \frac{1}{2}v$ so they are dependent



Independent Vectors

Definition: Independent Vectors

Two vectors u and v are **linearly independent** if both are nonzero and neither is a scalar multiple of the other.

- Example:

$$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

- It is impossible to find a constant c such that $u = cv$



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- Example:

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- It is impossible to find a constant c such that $u = cv$

Question

What linear combination of u and v will add up to zero?

$$\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



Definition: Linearly Independent

The vectors $v_1, \dots, v_n \in \mathbb{R}^m$ are **linearly independent** if

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$$

implies

$$\alpha_1 = \alpha_2 = \dots = \alpha_n = 0.$$

Otherwise, they are **linearly dependent**.

- Example:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

are independent

- None of the vectors can be built from a linear combination of the others.



Linear Independence and the Null Space

Definition: Nullity and Null Space

The **null space** $\text{null}(A)$ is the set of all vectors x such that $Ax = 0$.

The **nullity** $\text{nullity}(A)$ is the dimension of the null space.

- The following vectors are linearly independent:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

- Therefore, the only solution to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Key Idea

The columns of A are linearly independent if and only if $\text{null}(A) = \{0\}$, or equivalently, $\text{nullity}(A) = 0$.

Finding the Null Space

Example

Find the null space of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}.$$

Notice that the columns satisfy

$$a_{(3)} = a_{(1)} + a_{(2)} \quad \implies \quad a_{(1)} + a_{(2)} - a_{(3)} = 0.$$

Using the column view of matrix-vector multiplication,

$$A \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = a_{(1)} + a_{(2)} - a_{(3)} = 0.$$

Thus,

$$\text{null}(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}.$$



Linear Systems



Linear Systems as Matrix Equations

Here are three ways to express linear systems:

- Linear System

$$\begin{cases} 3x_2 + x_3 = 10 \\ x_1 - 2x_2 + x_3 = 6 \\ 3x_1 - x_3 = 13 \end{cases}$$

- Matrix form

$$\begin{bmatrix} 0 & 3 & 1 \\ 1 & -2 & 1 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \\ 13 \end{bmatrix}$$

- Compact notation

$$Ax = b$$



Matrix Inverse

Definition: Inverse

1. A square matrix $A \in \mathbb{R}^{n \times n}$ is **invertible** if there exists a matrix A^{-1} such that

$$A^{-1}A = AA^{-1} = I.$$

2. The **identity matrix** is a square matrix with ones on the main diagonal and zeros everywhere else:

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

3. The matrix A^{-1} is called the **inverse** of A .
4. A square matrix without an inverse is called **singular**.



Matrix Inverse

If A is invertible, then $Ax = b$ has the unique solution

$$x = A^{-1}b.$$

Proof:

$$Ax = b$$

$$A^{-1}Ax = A^{-1}b$$

$$Ix = A^{-1}b$$

$$x = A^{-1}b$$



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- In Matlab, `inv(A)` computes the inverse of A .
- This suggests computing the solution using $x = \text{inv}(A)*b$.



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- Explicitly computing the inverse is generally more expensive and can be less accurate than solving the system directly.

Goal

We want to solve $Ax = b$ without explicitly computing A^{-1} . A large portion of this class is about learning efficient methods for doing this.

Question

A grocery store sells apples at \$1 per pound and pineapples at \$3 per pound. You spend exactly \$40 and buy twice as much apple weight as pineapple weight.

- Is there any combination that works?
- If there is, is that the only combination?
- What if I give you \$100 instead?
- How does that affect the number of combinations?



Word Problem

Question

A grocery store sells apples at \$1 per pound and pineapples at \$3 per pound. You spend exactly \$40 and buy twice as much apple weight as pineapple weight.

- Is there any combination that works?
 - If there is, is that the only combination?
 - What if I give you \$100 instead?
 - How does that affect the number of combinations?
-
- Let a and p be the pounds of apples and pineapples purchased. Then

$$\begin{cases} a + 3p = 40, \\ a - 2p = 0. \end{cases}$$

- We want to investigate whether a solution exists and whether it is unique:

$$\begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ p \end{bmatrix} = \begin{bmatrix} 40 \\ 0 \end{bmatrix}.$$



Solvability



Singular Matrices

Definition:

A square matrix A is **nonsingular** (or **invertible**) if A^{-1} exists.

This definition of singularity has nothing to do with x or b

Examples of singular matrices:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$$

Determinant and Singularity

The determinant of a 2 by 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by the formula $\det(A) = ad - bc$. A square matrix is singular if and only if its determinant is zero.

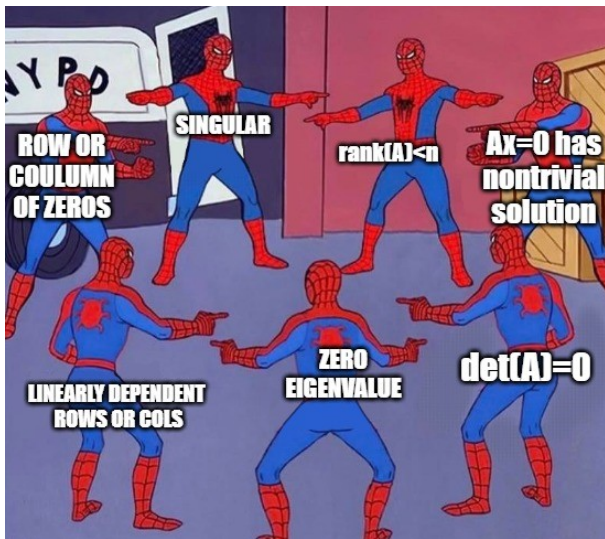
Singularity of A

For $A \in \mathbb{R}^{n \times n}$, the following conditions are equivalent:

- A is singular: A^{-1} does not exist
- The columns of A are linearly dependent
- The columns of A do not span $\mathbb{R}^n$
- The rows of A are linearly dependent
- $\text{rank}(A) < n$
- $\text{nullity}(A) > 0$
- $\det(A) = 0$
- $Ax = 0$ has a nonzero solution
- There exists $b \in \mathbb{R}^n$ for which $Ax = b$ has no solution
- A has an eigenvalue equal to zero
- The reduced row echelon form of A has at least one row of zeros



They're all equivalent



Putting it All Together

Problem to Solve

$$Ax = b$$

Homogeneous Problem

$$Ax = 0$$

Suppose $A \in \mathbb{R}^{n \times n}$ and the only solution to $Ax = 0$ is $x = 0$:

- The columns of A are linearly independent
- A^{-1} exists
- For every $b \in \mathbb{R}^n$, the system $Ax = b$ has the unique solution $x = A^{-1}b$



Rank-Nullity Theorem

Theorem

The Rank-Nullity theorem states: for any matrix $A \in \mathbb{R}^{m \times n}$,

$$\text{rank}(A) + \text{nullity}(A) = n.$$

- The rank and nullity add up to the number of columns of A .



Number of Solutions

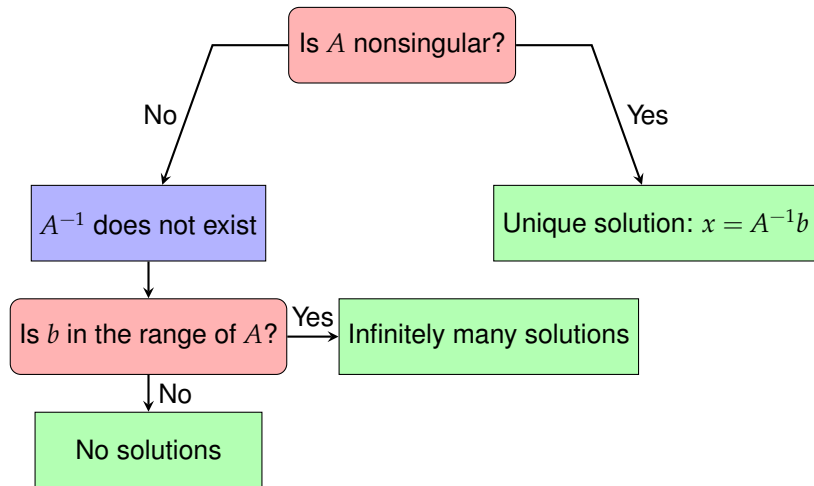
Consider $Ax = b$, where $A \in \mathbb{R}^{n \times n}$.

Three cases:

- 1 A is nonsingular: There is exactly one solution, $x = A^{-1}b$.
- 2 A is singular and $b \in \text{range}(A)$: There are infinitely many solutions.
- 3 A is singular and $b \notin \text{range}(A)$: There are no solutions.



Fundamental Flowchart



Example

Count the number of solutions to each linear system $Ax = b$:

① $A = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$

② $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

③ $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

① $\det(A) = 8 \neq 0$ so there's one solution.

② $\det(A) = 0$ but there are ∞ solutions $x = \begin{bmatrix} 1/2 \\ \alpha \end{bmatrix}$

③ $\det(A) = 0$ and $b \notin \text{range}(A)$ so no solutions



When b Is Not in the Range of A

Example

Determine whether b belongs to the range of A :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

- The 3rd column of A is redundant because

$$a_{(3)} = a_{(1)} + a_{(2)}$$

- We want to solve this system

$$\alpha \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- But the new matrix is not singular so $b \notin \text{range}(A)$



Revisit the Question

Question

A grocery store sells apples at \$1 per pound and pineapples at \$3 per pound. You spend exactly \$40 and buy twice as much apple weight as pineapple weight.

- Is there any combination that works?
- If there is, is that the only combination?
- What if I give you \$100 instead?
- How does that affect the number of combinations?

- System

$$\begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ p \end{bmatrix} = \begin{bmatrix} 40 \\ 0 \end{bmatrix}$$

- Since $\det(A) = -5 \neq 0$ one unique solution exists
- Changing the budget changes the solution, but not its existence or uniqueness:

$$\text{\$40 : } (a, p) = (16, 8), \quad \text{\$100 : } (a, p) = (40, 20).$$

