

3.6 - Impulsive Forces

(Section 3.4)

1. The Dirac Delta Function

1.1 Motivation

In a forced spring equation, the right-hand side represents an external force. We can use the Heaviside function to switch a force on or off at time $t = a$:

$$ax'' + bx' + cx = \underbrace{f(t)H(t-a)}_{\text{turn on at } t=a}, \quad \text{or} \quad ax'' + bx' + cx = \underbrace{f(t)[1 - H(t-a)]}_{\text{turn off at } t=a}.$$

What if an effect lasts only for a very short time? For example, striking a mass with a hammer, a brief burst of light, rapidly injecting a drug into the bloodstream. We idealize these brief events as **instantaneous impulses**. To represent them, we use the Dirac delta function $\delta(t)$:

$$ax'' + bx' + cx = \underbrace{\delta(t-a)}_{\text{impulse at } t=a}.$$

1.2 Definition of $\delta(t)$

The Dirac delta function $\delta(t)$ is defined in terms of rectangular pulses. Consider the parameterized rectangular function

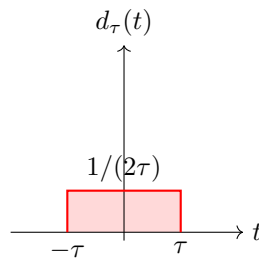
$$d_\tau(t) = \begin{cases} \frac{1}{2\tau}, & -\tau < t < \tau, \\ 0, & \text{otherwise,} \end{cases} \quad \text{where } \tau > 0.$$

When $d_\tau(t)$ is sketched, it appears as a rectangle with a width of 2τ and a height of $1/(2\tau)$, so its area is always one. That means the integral of d_τ is one as well:

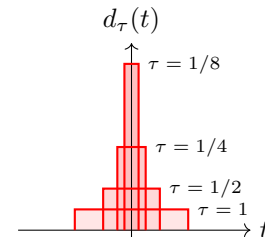
$$\text{Area} = \text{width} \times \text{height} = \int_{-\infty}^{\infty} d_\tau(t) dt = 1.$$

This is true for any value of τ . As $\tau \rightarrow 0^+$, the pulse becomes narrower and taller while retaining the same area. This approximates an impulse at $t = 0$. The **Dirac delta** function $\delta(t)$ is defined to be

$$\delta(t) = \lim_{\tau \rightarrow 0^+} d_\tau(t).$$



(a) A rectangular pulse of area 1.



(b) Narrower pulses have greater height.

Figure 1: The delta is an idealized limit of pulses with fixed area. This limit is understood through integration, rather than as an ordinary pointwise limit.

The delta function models an instantaneous impulse. Because its total area is 1, we call it a **unit impulse**. We can adjust its strength and timing:

$$f(t) = J \delta(t - a).$$

This represents an impulse of strength J applied at time $t = a$. In a spring-mass model, J is measured in newton-seconds (N·s). Its sign determines the direction of the impulse.

1.3 The Sifting Property

Theorem 1 (Sifting Property). *For a function f continuous at $t = a$,*

$$\int_{-\infty}^{\infty} f(t) \delta(t - a) dt = f(a).$$

The delta selects the value of f at the impulse time. If the integral is over a smaller interval, the same rule applies when a lies inside that interval. If a lies outside it, the integral is zero.

Proof. Assume f is continuous near a and approximate $\delta(t - a)$ by $d_{\tau}(t - a)$. Then

$$\int_{-\infty}^{\infty} f(t) d_{\tau}(t - a) dt = \frac{1}{2\tau} \int_{a-\tau}^{a+\tau} f(t) dt.$$

By the mean value theorem for integrals, this equals $f(c_{\tau})$ for some $c_{\tau} \in [a - \tau, a + \tau]$. As $\tau \rightarrow 0^+$, we have $c_{\tau} \rightarrow a$, so

$$\lim_{\tau \rightarrow 0^+} \int_{-\infty}^{\infty} f(t) d_{\tau}(t - a) dt = f(a).$$

This gives the sifting property. □

EXAMPLE 1

Evaluate:

$$(a) \int_{-\infty}^{\infty} \cos t \delta(t - \pi) dt, \quad (b) \int_{-\infty}^{\infty} t^2 \delta(t + 3) dt,$$

Solution.

(a) The impulse occurs at $t = \pi$, so

$$\int_{-\infty}^{\infty} \cos t \delta(t - \pi) dt = \cos \pi = \boxed{-1}.$$

(b) Write $\delta(t + 3) = \delta(t - (-3))$. The impulse occurs at $t = -3$:

$$\int_{-\infty}^{\infty} t^2 \delta(t + 3) dt = (-3)^2 = \boxed{9}.$$

1.4 Transform Formulas

Here is the formula for the Laplace transform of the delta function

$$\mathcal{L}[\delta(t)] = 1.$$

This is formula number 10 in the Laplace table of transforms. For $a > 0$, we have shifted delta function

$$\mathcal{L}[\delta(t-a)] = \int_0^\infty e^{-st} \delta(t-a) dt = e^{-as}.$$

This is formula number 11. We can use this to solve models with impulsive forcing.

2. Impulsive Forces

2.1 Underdamped Spring with an Impulse

Consider the spring-mass system

$$x'' + 2x' + 5x = 0, \quad x(0) = x'(0) = 0.$$

The characteristic roots are $\lambda = -1 \pm 2i$, so the system is underdamped. With zero initial displacement and velocity, its solution is $x(t) = 0$: the mass remains at rest. Now suppose the mass is struck impulsively at $t = 5$ in the positive direction. We model the impact by adding $\delta(t-5)$ to the right-hand side.

EXAMPLE 2

Solve the IVP using Laplace transforms:

$$x'' + 2x' + 5x = \delta(t-5), \quad x(0) = x'(0) = 0.$$

Write the solution in Heaviside and piecewise forms.

Solution.

Step 1. Transform and solve for $X(s)$. Using the zero initial conditions,

$$(s^2 + 2s + 5)X(s) = e^{-5s}.$$

Complete the square to obtain

$$X(s) = e^{-5s} \underbrace{\frac{1}{(s+1)^2 + 4}}_{G(s)}.$$

Step 2. Invert the rational part. The transform contains both an exponential shift and a time delay:

$$X(s) = \underbrace{e^{-5s}}_{\text{time delay}} \underbrace{\frac{1}{(s+1)^2 + 4}}_{\text{exponential shift}}.$$

Set e^{-5s} aside. Since

$$\mathcal{L}^{-1}\left[\frac{1}{s^2 + 4}\right] = \frac{1}{2} \sin 2t,$$

the exponential shift gives

$$g(t) = \mathcal{L}^{-1}[G(s)] = \frac{1}{2} e^{-t} \sin 2t.$$

Step 3. Apply the delay. Delay the entire function $g(t)$ by 5:

$$x(t) = H(t-5)g(t-5).$$

Therefore,

$$x(t) = \frac{1}{2}H(t-5)e^{-(t-5)}\sin(2(t-5)).$$

In piecewise form,

$$x(t) = \begin{cases} 0, & 0 \leq t < 5, \\ \frac{1}{2}e^{-(t-5)}\sin(2(t-5)), & t \geq 5. \end{cases}$$

2.2 Position and Velocity at Impact

We use the notations $x(a^+)$ and $x(a^-)$ to denote one sided limits from the right and left sides, respectively. These are useful for describing jumps. Adding the forcing term $\delta(t-5)$ creates a jump in the velocity x' , but not in the position x at $t=5$:

$$x(5^-) = x(5^+) = 0, \quad x'(5^-) = 0, \quad x'(5^+) = 1.$$

This makes sense since the mass cannot magically teleport after being struck. More generally,

$$mx'' + \gamma x' + kx = J\delta(t-a) \implies x'(a^+) - x'(a^-) = \frac{J}{m}.$$

A positive impulse increases the velocity; a negative impulse decreases it. This gives us a hint on how to plot the solution.

2.3 Sketching the Solution

To sketch our solution:

1. Before $t=5$, draw $x=0$.
2. At $t=5$, keep the curve connected but change its slope from 0 to 1.
3. After $t=5$, draw a decaying oscillation with period π .

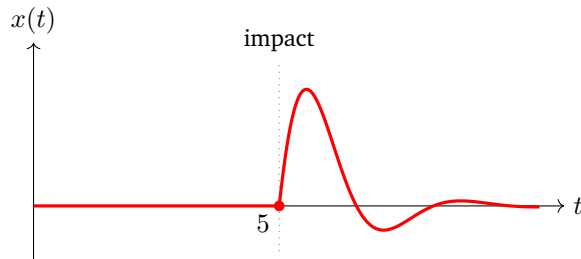


Figure 2: The mass begins oscillating after the impulse. Position is continuous, but its slope jumps at $t=5$.

In general, a spring may receive multiple impulses:

$$mx'' + \gamma x' + kx = J_1\delta(t-a_1) + J_2\delta(t-a_2).$$

The solution has the form

$$x(t) = h(t) + J_1H(t-a_1)g(t-a_1) + J_2H(t-a_2)g(t-a_2),$$

where $h(t)$ describes the motion from the initial conditions and $g(t)$ is the response to a unit impulse applied at time zero. At each impact, the position stays continuous and the velocity changes by J_i/m . Depending on its direction relative to the motion, an impulse can speed up, slow down, or reverse the mass.