

3.5 - Piecewise Forcing and Convolution

(Section 3.2, 3.3)

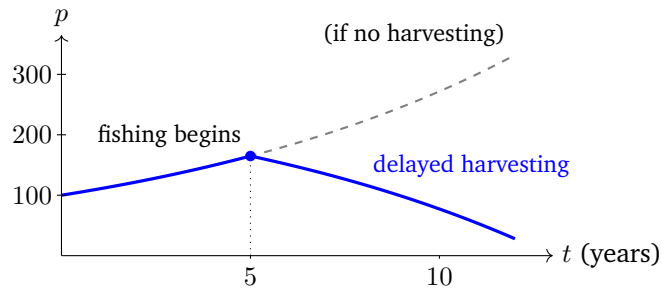
1. First-Order with Piecewise Forcing

1.1 Returning to the Harvesting Model

Last lecture, we introduced a fish farm containing 100 fish. After five years, harvesting begins at 30 fish per year:

$$p' = 0.1p - 30H(t - 5), \quad p(0) = 100.$$

The continuous solution is plotted below. We will solve this IVP in two ways: separately before and after the switch, then using Laplace transforms.



1.2 Solving in Pieces

We split the IVP into two sub-problems. Let $p_1(t)$ describe the population before harvesting begins, and let $p_2(t)$ describe the population afterward. Our goal is to find both functions and combine them:

$$p(t) = \begin{cases} p_1(t), & 0 \leq t < 5, \\ p_2(t), & t \geq 5. \end{cases}$$

The functions p_1 and p_2 are solutions to the sub-problems:

$$\begin{cases} p_1' = 0.1p_1, \\ p_1(0) = 100. \end{cases} \quad \begin{cases} p_2' = 0.1p_2 - 30, \\ p_2(5) = k. \end{cases}$$

There are two key differences in the sub-problems. First, the DEs are different because of the Heaviside term $f = -30H(t - 5)$. Second, the initial conditions are different. The condition $p_1(0) = 100$ is a given condition. The IC for p_2 begins at $t = 5$ and the value k must be determined after solving for p_1 . The value for k is

$$k = p_1(5).$$

This condition ensures that piecewise function $p(t)$ is continuous at $t = 5$.

EXAMPLE 1

Solve the IVP in pieces using the method of undetermined coefficients:

$$p' = 0.1p - 30H(t - 5), \quad p(0) = 100.$$

Solution.

Step 1. Solve for p_1 . Before harvesting,

$$p_1' = 0.1p_1, \quad p_1(0) = 100,$$

so

$$p_1(t) = 100e^{0.1t}, \quad p_1(5) = 100e^{1/2}.$$

Step 2. Solve for p_2 . The second sub-problem is

$$p_2' - 0.1p_2 = -30, \quad p_2(5) = 100e^{1/2}.$$

The homogeneous solution is $Ce^{0.1t}$. Trying a constant particular solution $p_p(t) = A$ gives $-0.1A = -30$, so $A = 300$. Thus, the general solution is

$$p_2(t) = Ce^{0.1t} + 300.$$

Applying the initial condition,

$$p_2(5) = Ce^{1/2} + 300 \equiv 100e^{1/2} \implies C = 100 - 300e^{-1/2}.$$

Therefore,

$$p_2(t) = 100e^{0.1t} + 300(1 - e^{0.1(t-5)}).$$

Step 3. Combine p_1 and p_2 . Use each solution on its corresponding interval:

$$p(t) = \begin{cases} 100e^{0.1t}, & 0 \leq t < 5, \\ 100e^{0.1t} + 300(1 - e^{0.1(t-5)}), & t \geq 5. \end{cases}$$

1.3 Solving with Laplace Transforms

Laplace transforms let us solve the IVP without splitting it into sub-problems. We obtain an answer in Heaviside notation, then rewrite it piecewise.

EXAMPLE 2

Solve the same IVP using Laplace transforms:

$$p' = 0.1p - 30H(t-5), \quad p(0) = 100.$$

Write the answer using Heaviside notation and in piecewise form.

Solution.

Step 1. Transform and solve for $P(s)$. Let $P(s) = \mathcal{L}[p(t)]$. Taking transforms gives

$$sP(s) - 100 = 0.1P(s) - 30\frac{e^{-5s}}{s}.$$

Solving for $P(s)$,

$$P(s) = \frac{100}{s - 0.1} - 30e^{-5s} \underbrace{\frac{1}{s(s - 0.1)}}_{G(s)}.$$

Step 2. Find the inverse of the rational part. Ignore the $-30e^{-5s}$ factor for the time being, and focus on the rational factor $G(s)$. Using partial fractions,

$$G(s) = -\frac{10}{s} + \frac{10}{s - 0.1},$$

so the inverse transform is

$$g(t) = \mathcal{L}^{-1}[G(s)] = 10(e^{0.1t} - 1).$$

Step 3. Find the full inverse. The switching property gives

$$p(t) = 100e^{0.1t} - 30H(t - 5)g(t - 5).$$

Substituting $g(t - 5)$,

$$p(t) = 100e^{0.1t} + 300H(t - 5)(1 - e^{0.1(t-5)}).$$

Setting $H(t - 5) = 0$ before the switch and 1 afterward,

$$p(t) = \begin{cases} 100e^{0.1t}, & 0 \leq t < 5, \\ 100e^{0.1t} + 300(1 - e^{0.1(t-5)}), & t \geq 5. \end{cases}$$

This agrees with the solution obtained in pieces.

2. Second-Order with Piecewise Forcing

2.1 A Delayed Forcing Term

The Laplace transform method also applies to second-order equations with piecewise forcing. Consider the problem

$$x'' - x = g(t), \quad x(0) = 1, \quad x'(0) = 0,$$

where

$$g(t) = \begin{cases} 0, & 0 \leq t < 2, \\ 2 \cos(t - 2), & t \geq 2. \end{cases}$$

The first step to solving this equation is to rewrite $f(t)$ using the Heaviside function:

$$g(t) = 2 \cos(t - 2)H(t - 2).$$

We will need to transform the forcing term as well

$$\mathcal{L}[g(t)] = e^{-as}F(s) = e^{-2s} \frac{2s}{s^2 + 1}.$$

2.2 Solving with Laplace Transforms

Follow the 3 steps to solve IVPs involving piecewise functions

1. Transform the entire DE and solve for $X(s)$.
2. Invert the rational part.
3. Find the full inverse.

If asked, you should be able to rewrite the solution in piecewise form.

EXAMPLE 3

Solve the IVP using Laplace transforms:

$$x'' - x = \begin{cases} 0, & 0 \leq t < 2, \\ 2 \cos(t - 2), & t \geq 2, \end{cases} \quad x(0) = 1, \quad x'(0) = 0.$$

Write the solution using Heaviside notation and in piecewise form.

Solution.

Step 1. Transform the entire DE and solve for X .

$$\begin{aligned} \mathcal{L}[x'' - x] &= \mathcal{L}[2 \cos(t - 2)H(t - 2)] \\ s^2 X(s) - sx(0) - x'(0) - X(s) &= e^{-2s} \frac{2s}{s^2 + 1} \\ s^2 X(s) - s - X(s) &= e^{-2s} \frac{2s}{s^2 + 1} \\ (s^2 - 1)X(s) &= s + e^{-2s} \frac{2s}{s^2 + 1} \\ X(s) &= \underbrace{\frac{s}{s^2 - 1}}_{X_1(s)} + e^{-2s} \underbrace{\frac{2s}{(s^2 - 1)(s^2 + 1)}}_{X_2(s)}. \end{aligned}$$

Ignore e^{-2s} for now, and focus on X_1 and X_2 .

Step 2. Invert the rational part. I'll leave it as an exercise for you to verify the following partial fraction decomposition

$$X_1(s) = \frac{1}{2(s-1)} + \frac{1}{2(s+1)},$$

and so

$$X_2(s) = \frac{s}{s^2 - 1} - \frac{s}{s^2 + 1} = \frac{1}{2(s-1)} + \frac{1}{2(s+1)} - \frac{s}{s^2 + 1}.$$

Therefore, the base inverse transforms are

$$x_1(t) = \mathcal{L}^{-1}[X_1(s)] = \frac{1}{2}e^t + \frac{1}{2}e^{-t},$$

$$x_2(t) = \mathcal{L}^{-1}[X_2(s)] = \frac{1}{2}e^t + \frac{1}{2}e^{-t} - \cos t.$$

Step 3. Find the full inverse. At this point, bring e^{-2s} back into the picture. We need to utilize the switch property to find the full inverse:

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}[X_1(s) + e^{-2s}X_2(s)] \\ &= x_1(t) + H(t-2)x_2(t-2) \\ &= \boxed{\frac{1}{2}e^t + \frac{1}{2}e^{-t} + H(t-2) \left(\frac{1}{2}e^{(t-2)} + \frac{1}{2}e^{-(t-2)} - \cos(t-2) \right)} \end{aligned}$$

2.3 Writing Solutions Compactly

The **hyperbolic functions**

$$\cosh t = \frac{e^t + e^{-t}}{2}, \quad \sinh t = \frac{e^t - e^{-t}}{2}$$

provide convenient shorthand for pairs of exponentials. For example, when $k > 0$,

$$x'' - k^2 x = 0 \implies x(t) = c_1 \cosh(kt) + c_2 \sinh(kt).$$

Using this notation, the solution to Example 3 becomes

$$x(t) = \cosh t + H(t-2)[\cosh(t-2) - \cos(t-2)].$$

More generally, we can name repeated expressions ourselves. Define

$$h(t) = \cosh t, \quad g(t) = \cosh t - \cos t.$$

Then both the Heaviside and piecewise forms are compact:

$$x(t) = h(t) + H(t-2)g(t-2) = \begin{cases} h(t), & 0 \leq t < 2, \\ h(t) + g(t-2), & t \geq 2. \end{cases}$$

Naming functions is a useful way to simplify long answers, provided those functions are explicitly defined.

3. Convolution

3.1 Inverting a Product

So far, we have used partial fractions to rewrite a transform as a sum of familiar terms. For example,

$$X(s) = \frac{1}{s(s^2 + 4)}$$

can be decomposed and inverted term by term. But this expression is also a product

$$X(s) = \underbrace{\frac{1}{s}}_{F(s)} \underbrace{\frac{1}{s^2 + 4}}_{G(s)}.$$

We know the inverse transform of each factor. Can we combine them to find the inverse of the product?

Simply multiplying the inverse transforms does not work:

$$\mathcal{L}^{-1}[F(s)G(s)] \neq f(t)g(t) \quad \text{in general.}$$

Instead, we need a new operation called **convolution**.

3.2 The Convolution Integral

The **convolution** of two functions f and g is defined by

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau) d\tau.$$

The symbol $*$ denotes convolution, not ordinary multiplication. Inside the integral, t is treated as a constant and τ is the variable of integration. After integrating and evaluating the endpoints, the result is a function of t .

Convolution is commutative:

$$f * g = g * f.$$

Thus, we may also write

$$(f * g)(t) = \int_0^t f(t - \tau)g(\tau) d\tau = \int_0^t f(\tau)g(t - \tau) d\tau.$$

Choose whichever form makes the integration easier.

3.3 The Convolution Formula

If $\mathcal{L}[f(t)] = F(s)$ and $\mathcal{L}[g(t)] = G(s)$, then, wherever the transforms converge absolutely,

$$\boxed{\mathcal{L}[(f * g)(t)] = F(s)G(s).}$$

This formula is more commonly used in reverse

$$\boxed{\mathcal{L}^{-1}[F(s)G(s)] = \int_0^t f(t - \tau)g(\tau) d\tau.}$$

To use this formula:

1. Split the transform into two familiar factors.
2. Find their inverse transforms.
3. Form the convolution integral and evaluate it.

EXAMPLE 4

Use convolution to find

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2 + 4)}\right].$$

Do not leave your answer in terms of integrals.

Solution.

Step 1. Identify the two factors. Write

$$\frac{1}{s(s^2 + 4)} = \underbrace{\frac{1}{s}}_{F(s)} \underbrace{\frac{1}{s^2 + 4}}_{G(s)}.$$

Step 2. Find the inverse transforms. From the transform table,

$$f(t) = 1, \quad g(t) = \frac{1}{2} \sin 2t.$$

Step 3. Evaluate the convolution integral. Using $f(t - \tau) = 1$ and $g(\tau) = \frac{1}{2} \sin 2\tau$,

$$\begin{aligned} x(t) &= \int_0^t f(t - \tau)g(\tau) d\tau \\ &= \frac{1}{2} \int_0^t \sin 2\tau d\tau \\ &= \left[-\frac{1}{4} \cos 2\tau \right]_0^t = \boxed{\frac{1}{4}(1 - \cos 2t).} \end{aligned}$$

3.4 Forcing beyond Undetermined Coefficients

The method of undetermined coefficients works for polynomials, exponentials, sines, cosines, and their sums and products. But what if the forcing is e^{-t^2} ?

Repeated differentiation produces polynomials of increasing degree multiplied by e^{-t^2} , so the usual finite trial form does not work. Convolution lets us write a solution without choosing a trial function or even finding the forcing's Laplace transform explicitly.

EXAMPLE 5

Solve the IVP

$$x'' + 9x = e^{-t^2}, \quad x(0) = 1, \quad x'(0) = 0.$$

You may leave the answer as a convolution integral.

Solution.

Step 1. Transform the DE. Let $X(s) = \mathcal{L}[x(t)]$ and denote the transform of the forcing by

$$F(s) = \mathcal{L}[e^{-t^2}].$$

We do not need an explicit formula for $F(s)$. Taking transforms and using the initial conditions gives

$$s^2 X(s) - s + 9X(s) = F(s).$$

Step 2. Solve for $X(s)$.

$$X(s) = \frac{s}{s^2 + 9} + \underbrace{\frac{1}{s^2 + 9}}_{G(s)} F(s).$$

Step 3. Invert using convolution. The first term has inverse $\cos 3t$. For the product, we know both base functions:

$$g(t) = \mathcal{L}^{-1}[G(s)] = \frac{1}{3} \sin 3t, \quad f(t) = e^{-t^2}.$$

By the convolution theorem,

$$\mathcal{L}^{-1}[G(s)F(s)] = \int_0^t g(t - \tau)f(\tau) d\tau.$$

Therefore,

$$x(t) = \cos 3t + \frac{1}{3} \int_0^t \sin(3(t - \tau))e^{-\tau^2} d\tau.$$

This is an exact solution. The integral does not need an elementary antiderivative; it can also be evaluated numerically to calculate or plot $x(t)$.