

3.4 - Heaviside functions, Piece-wise Functions

(Section 3.1)

1. Motivation for Discontinuous Functions

1.1 Modeling Delayed Effects

Suppose a fish farm initially contains 100 fish. Without harvesting, the population grows according to

$$P' = 0.1P, \quad P(0) = 100,$$

so the fish population is $P(t) = 100e^{0.1t}$, where t is measured in years. Now suppose we want to introduce a delayed effect into our model. Let's say after waiting five years, the farmers begin removing 30 fish per year. We can model the fish population using a *discontinuous function*:

$$P' = 0.1P + f(t), \quad P(0) = 100,$$

where

$$f(t) = \begin{cases} 0, & 0 \leq t < 5, \\ -30, & t \geq 5. \end{cases}$$

1.2 Delayed Harvesting

The function $f(t)$ represents *delayed harvesting*. Delayed harvesting introduces a shift in the dynamics at future time. Even when $f(t)$ is discontinuous the solution $P(t)$ is actually continuous as shown in the figure below. This makes logical sense because the fish are not magically vanishing; they're being farmed. We could solve this problem using undetermined coefficients separately before and after $t = 5$, but we would like a method that handles the entire problem at once.

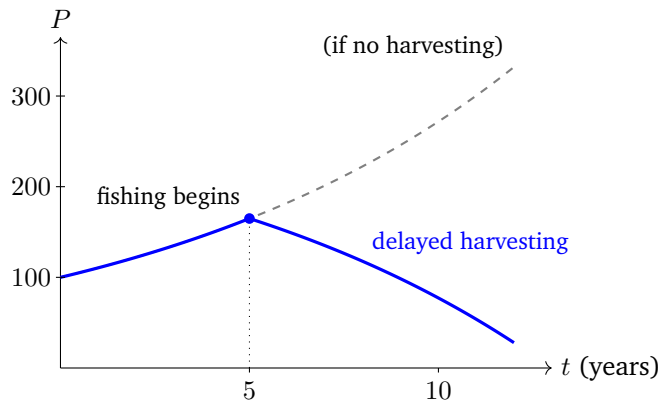


Figure 1: The blue curve $P(t)$ represents the fish population over 12 years. Fish population is on an upward trend until $t = 5$ when the harvesting begins.

1.3 Goal

More generally, our goal is to solve equations of the form

$$ax'' + bx' + cx = f(t),$$

where the forcing term $f(t)$ is *piecewise* and potentially *discontinuous*. Laplace transforms are particularly useful for these problems. However, we will not be solving this equation today. We will focus on how to find transforms and inverse transforms for piecewise functions.

2. Heaviside Function and its Uses

2.1 Definition

The **Heaviside function**, or **unit step function**, is defined by

$$H(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$

It is often useful to translate the Heaviside along the t -axis. For example, if we wanted to shift $H(t)$ to the right by 4 units, we would write

$$H(t-4) = \begin{cases} 0, & t-4 < 0, \\ 1, & t-4 \geq 0. \end{cases} \implies H(t-4) = \begin{cases} 0, & t < 4, \\ 1, & t \geq 4. \end{cases}$$

Think of $H(t-4)$ as a switch that is initially turned off and then switches on at $t = 4$. The sketches of $H(t)$ and $H(t-4)$ are plotted below. Use open and filled circles at the jumps to indicate the value of the function at the discontinuity.

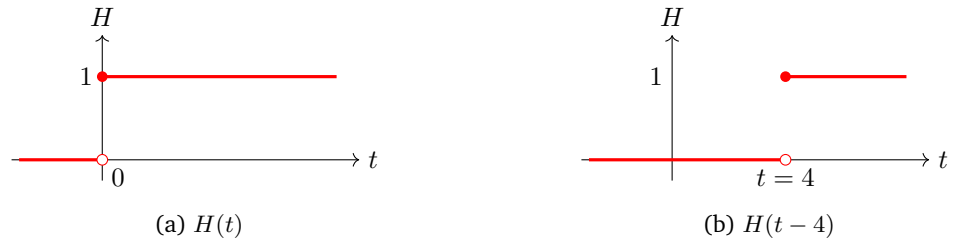


Figure 2: The shift 4 controls the time at which the jump occurs.

Using this simple function, we may construct an **in-line representation** of piecewise functions. For example, the delayed harvesting function

$$f(t) = \begin{cases} 0, & 0 \leq t < 5, \\ -30, & t \geq 5. \end{cases}$$

can be written in one line as $f(t) = -30H(t-5)$.

2.2 Suppressing vs. Delaying

Suppose $f(t)$ models an external force. This may be the forcing function for a differential equation. Multiplying a function $f(t)$ by a Heaviside function has some interesting properties.

1. Multiplying $f(t)$ by $H(t-a)$ yields

$$f(t)H(t-a) = \begin{cases} 0, & t < a, \\ f(t), & t \geq a. \end{cases}$$

The product is zero until time $t = a$. This results in *suppressing* the effect of $f(t)$ on a system until time $t = a$.

2. Now suppose $f(t)$ changes with time. Multiplying $f(t-a)$ by $H(t-a)$ yields

$$f(t-a)H(t-a) = \begin{cases} 0, & t < a, \\ f(t-a), & t \geq a. \end{cases}$$

At $t = a$, the delayed function begins with the value $f(0)$. This results in *delaying* the start of $f(t)$ until time $t = a$.

The following example illustrates the graphical difference between the two effects.

EXAMPLE 1

Let $f(t) = t$. Sketch and compare

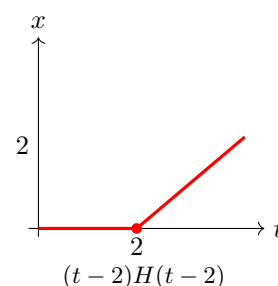
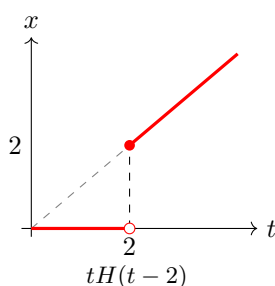
$$tH(t-2) \quad \text{and} \quad (t-2)H(t-2).$$

Solution.

Write each function piecewise:

$$tH(t-2) = \begin{cases} 0, & t < 2, \\ t, & t \geq 2, \end{cases} \quad (t-2)H(t-2) = \begin{cases} 0, & t < 2, \\ t-2, & t \geq 2. \end{cases}$$

The first function switches on at height 2. The second starts at height 0 and follows the original ramp with a delay of two units.

**2.3 Jump from Constant to Constant**

Consider a function that jumps from one constant to another constant at time $t = a$. For example,

$$f(t) = \begin{cases} L_1, & t < a \\ L_2, & t \geq a. \end{cases}$$

This function can be written compactly in one line using the Heaviside function

$$f(t) = L_1 + (L_2 - L_1)H(t - a).$$

The function starts off at level L_1 . Then at time $t = a$, the difference $L_2 - L_1$ is added to simultaneously remove L_1 and introduce L_2 .

EXAMPLE 2

Sketch the function and write it using Heaviside functions:

$$f(t) = \begin{cases} 3, & 0 \leq t < 2, \\ 5, & 2 \leq t < 3, \\ -1, & t \geq 3, \end{cases}$$

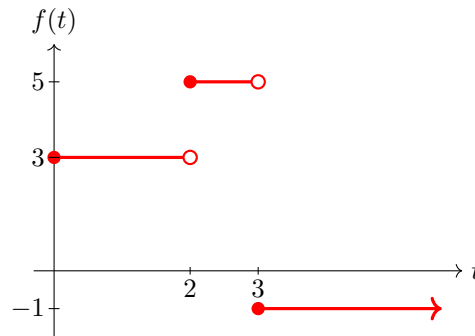
Solution.

Start at 3. The successive changes are

Time	Change in value	Term to add
0	3	
2	$5 - 3 = 2$	$2H(t - 2)$
3	$-1 - 5 = -6$	$-6H(t - 3)$

Therefore, $f(t)$ can be written as

$$f(t) = 3 + 2H(t - 2) - 6H(t - 3).$$



2.4 Jump from Function to Function

The same method works when the pieces are functions:

$$f(t) = \begin{cases} g(t), & 0 \leq t < a, \\ h(t), & t \geq a. \end{cases}$$

Start with $g(t)$, then at $t = a$ simultaneously add $h(t)$ and subtract $g(t)$:

$$f(t) = g(t) + (h(t) - g(t))H(t - a).$$

Before a , the added term is zero. After a , the expression simplifies to $g(t) + h(t) - g(t) = h(t)$.

EXAMPLE 3

Write the following function using a Heaviside function:

$$f(t) = \begin{cases} 1, & 0 \leq t < 1, \\ t^2, & t \geq 1. \end{cases}$$

Solution.

The initial expression is 1, and the new expression is t^2 . Thus,

$$f(t) = 1 + (t^2 - 1)H(t - 1).$$

3. Transform Formulas

3.1 Transform of the Heaviside Function

Here is the first transform formula involving the step functions. For $a \geq 0$,

$$\mathcal{L}[H(t - a)] = \frac{e^{-as}}{s}, \quad s > 0.$$

This is formula number 8 in the Laplace table. The proof of this formula will be a homework exercise. This formula allows us to find transforms of some piecewise functions. Take the

harvesting function, for example

$$f(t) = \begin{cases} 0, & 0 \leq t < 5, \\ -30, & t \geq 5. \end{cases}$$

To find the transform of $f(t)$, you must first rewrite it using the step function

$$f(t) = -30H(t - 5).$$

Then we can apply the new formula

$$\mathcal{L}[f(t)] = \mathcal{L}[-30H(t - 5)] = -30 \frac{e^{-5s}}{s}.$$

3.2 Transform of a Delayed Function

The product $H(t - a)f(t - a)$ has the effect of delaying the start of $f(t)$ until $t = a$. There is a formula for the transform of this product. Let $\mathcal{L}[f(t)] = F(s)$ and $a \geq 0$. Then

$$\boxed{\mathcal{L}[H(t - a)f(t - a)] = e^{-as}F(s).}$$

This is formula 9a in the Laplace table. The textbook calls this the **switching property**. The proof of this formula is left as a homework exercise. Here is an example of how to utilize this formula.

EXAMPLE 4

Find the transform

$$\mathcal{L}[(t - 3)^2 H(t - 3)].$$

Solution.

Step 1. Write down the formula.

$$\mathcal{L}[(t - 3)^2 H(t - 3)] = \mathcal{L}[H(t - a)f(t - a)] = e^{-as}F(s).$$

Step 2. Identify the shift a and the base transform. By comparison,

$$\mathcal{L}[H(t - 3)(t - 3)^2] = \mathcal{L}[H(t - a)f(t - a)]$$

we see that the shift is $a = 3$ and the base transform is

$$f(t) = t^2, \quad F(s) = \mathcal{L}[t^2] = \frac{2}{s^3}.$$

Step 3. Substitute.

$$\mathcal{L}[(t - 3)^2 H(t - 3)] = e^{-as}F(s) = e^{-3s}F(s) = \boxed{\frac{2e^{-3s}}{s^3}}.$$

3.3 When Shifts Do Not Match

In the previous example,

$$x(t) = \underbrace{(t - 3)^2}_{\text{shift 3}} \underbrace{H(t - 3)}_{\text{shift 3}}$$

the shifts matched. In some functions, such as

$$x(t) = \underbrace{e^{3t}}_{\text{shift 0}} \underbrace{H(t-2)}_{\text{shift 2}}$$

the shifts do not match. There are two options to deal with this:

Option 1: Rewrite the exponential so that its shift matches the step function:

$$\begin{aligned} e^{3t}H(t-2) &= e^{3(t-2+2)}H(t-2) \\ &= e^6 e^{3(t-2)}H(t-2). \end{aligned}$$

The base function is $f(t) = e^{3t}$, with transform $F(s) = 1/(s-3)$. Applying the switching property,

$$\begin{aligned} \mathcal{L}[e^{3t}H(t-2)] &= e^6 e^{-2s} \mathcal{L}[e^{3t}] \\ &= \boxed{\frac{e^{-2s+6}}{s-3}}. \end{aligned}$$

Option 2: Since this occurs frequently, you can use this convenient formula

$$\boxed{\mathcal{L}[f(t)H(t-a)] = e^{-as} \mathcal{L}[f(t+a)]}$$

This is formula 9b in the Laplace table. The following example demonstrates how this works.

EXAMPLE 5

Find

$$\mathcal{L}[e^{3t}H(t-2)].$$

Solution.

Here, $f(t) = e^{3t}$ and $a = 2$. Using formula 9b,

$$\begin{aligned} \mathcal{L}[e^{3t}H(t-2)] &= e^{-2s} \mathcal{L}[e^{3(t+2)}] \\ &= e^{-2s} e^6 \mathcal{L}[e^{3t}] \\ &= \boxed{\frac{e^{-2s+6}}{s-3}}. \end{aligned}$$

4. Inverse Transforms

4.1 Switching Property in Reverse

Since the goal is to solve differential equations with Heaviside forcing, we must learn how to use the switching property in reverse. The inverse form of the switching property is

$$\boxed{\mathcal{L}^{-1}[e^{-as}F(s)] = H(t-a)f(t-a)}$$

Use the following steps:

1. Write the formula.
2. Identify the shift a and the base transform $f(t) \rightarrow F(s)$.
3. Substitute

The result is zero before the delay and follows $f(t-a)$ afterward.

EXAMPLE 6

Find the inverse transform and write the answer piecewise:

$$\mathcal{L}^{-1} \left[e^{-2s} \frac{2}{s^2 + 49} \right].$$

Solution.

Step 1. Write down the formula.

$$\mathcal{L}^{-1} \left[e^{-2s} \frac{2}{s^2 + 49} \right] = \mathcal{L}^{-1}[e^{-as} F(s)] = H(t - a)f(t - a).$$

Step 2. Identify the delay a and the base function. By comparison,

$$a = 2, \quad F(s) = \frac{2}{s^2 + 49} \implies f(t) = \mathcal{L}^{-1} \left[\frac{2}{s^2 + 49} \right] = \frac{2}{7} \sin 7t.$$

Step 3. Substitute.

$$x(t) = H(t - 2)f(t - 2) = \boxed{\frac{2}{7} \sin(7(t - 2))H(t - 2)}.$$

4.2 Heaviside to Piecewise

The advantage of Heaviside notation is its compactness. The solution to Example 6 can be written in a single line

$$x(t) = \frac{2}{7} \sin(7(t - 2))H(t - 2).$$

But this can also be a downside since it is difficult to interpret. The function is easier to conceptualize when rewritten as a piecewise function:

$$x(t) = \begin{cases} 0, & 0 \leq t < 2, \\ \frac{2}{7} \sin(7(t - 2)), & t \geq 2. \end{cases}$$

4.3 Moving Forward

You are now ready to solve the IVP

$$P' = 0.1P + f(t), \quad P(0) = 100,$$

where

$$f(t) = \begin{cases} 0, & 0 \leq t < 5, \\ -30, & t \geq 5. \end{cases}$$

We will do this in the next lecture. Naturally, we solve this equation with Laplace transforms. However, the solution $P(t)$ will be in terms of the Heaviside function. You will change it to piecewise form for interpretation.