

## 3.2 - The Shift Formula

(Section 3.1)

### 1. Need More Transform Formulas

#### 1.1 What We Can Solve

Our goal is to use Laplace transforms to solve more types of second-order DEs. Our current table lets us recover constants, polynomials, exponentials, sines, and cosines.

If the solution to the DE is a sum of exponentials

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \implies X = \frac{As + B}{(s - \lambda_1)(s - \lambda_2)}$$

then we can use partial fraction decomposition on  $X(s)$ . Doing so will yield sums of formulas that appear in the right column of the transform table.

The same is true for purely imaginary eigenvalues  $\lambda = \pm \beta i$ . The solution is a sum of sine and cosine terms

$$x = c_1 \cos \beta t + c_2 \sin \beta t \implies X = \frac{As + B}{s^2 + \beta^2}$$

| Current Table of Transforms |                 |                               |
|-----------------------------|-----------------|-------------------------------|
|                             | $x(t)$          | $X(s)$                        |
| 1                           | 1               | $\frac{1}{s}$                 |
| 2                           | $t$             | $\frac{1}{s^2}$               |
| 3                           | $t^n$           | $\frac{n!}{s^{n+1}}$          |
| 4                           | $e^{at}$        | $\frac{1}{s-a}$               |
| 5                           | $\cos(\beta t)$ | $\frac{s}{s^2 + \beta^2}$     |
| 6                           | $\sin(\beta t)$ | $\frac{\beta}{s^2 + \beta^2}$ |

The transforms of these terms also appear in the table. However, other root types produce functions that our table does not yet include.

#### 1.2 What We Cannot Solve Yet

Now consider an equation with a repeated root:

$$x'' + 4x' + 4x = 0.$$

which has the general solution

$$x(t) = c_1 e^{-2t} + c_2 t e^{-2t}.$$

| Root type                   | Example              | Can we solve with Laplace? |
|-----------------------------|----------------------|----------------------------|
| Distinct real               | $x'' + 3x' + 2x = 0$ | ✓                          |
| Purely imaginary            | $x'' + 4x = 0$       | ✓                          |
| Repeated real               | $x'' + 4x' + 4x = 0$ |                            |
| Complex ( $\alpha \neq 0$ ) | $x'' + 2x' + 5x = 0$ |                            |

We know the transforms of  $t$  and  $e^{-2t}$  separately, but our table does not yet include their product  $t e^{-2t}$ . Similarly, consider an equation with complex roots having a nonzero real part:

$$x'' + 2x' + 5x = 0.$$

The roots are  $\lambda = -1 \pm 2i$ , so we expect

$$x(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t).$$

Our *current table of transforms* does not have a formula for the transform of this solution.

#### 1.3 Goal

Notice that both the repeated roots and complex roots cases have a solution of the form

$$e^{at} f(t).$$

This is the missing piece. The goal of today is to learn the *exponential shift property* and solve the 2nd order DEs with repeated or complex roots.

## 2. The Exponential Shift Property

### 2.1 The Shift Property

Multiplying a function by an exponential produces a shift in the Laplace domain.

**Theorem 1** (Shift Property). *If*

$$\mathcal{L}[f(t)] = F(s),$$

*then*

$$\mathcal{L}[e^{at}f(t)] = F(s - a).$$

This will be given as Formula 7 in the Laplace table. Although proofs are not usually required, the calculation below illustrates a technique you will use in the homework. We therefore present it as a worked example.

#### EXAMPLE 1

Prove the formula

$$\mathcal{L}[e^{at}f(t)] = F(s - a).$$

**Solution.**

By definition

$$\mathcal{L}[e^{at}f(t)] = \int_0^{\infty} e^{at}f(t)e^{-st} dt.$$

We can manipulate the exponent and create a new variable  $s^* = s - a$ :

$$\begin{aligned} \mathcal{L}[e^{at}f(t)] &= \int_0^{\infty} e^{at}f(t)e^{-st} dt && \text{(by definition)} \\ &= \int_0^{\infty} f(t)e^{-t(s-a)} dt \\ &= \int_0^{\infty} f(t)e^{-ts^*} dt && \text{(let } s^* = s - a) \\ &= F(s^*) && \text{(by definition)} \end{aligned}$$

Therefore, we have

$$\mathcal{L}[e^{at}f(t)] = F(s^*) = F(s - a).$$

The shift property lets us build new transform formulas from ones already in our table. First, identify the *base transform*  $f(t) \rightarrow F(s)$ . Then account for the exponential factor  $e^{at}$  by replacing  $s$  with  $s - a$ . Let's apply this to  $te^{-2t}$ , the term that appeared in our repeated-root solution.

#### EXAMPLE 2

Compute the transform

$$\mathcal{L}[te^{-2t}].$$

**Solution.**

First, identify the *base transform*. Since our function is  $t$  times  $e^{-2t}$  the base function is  $f(t) = t$ . Find the transform of  $f(t)$  by itself

$$\mathcal{L}[f(t)] = \mathcal{L}[t] = \frac{1}{s^2} =: F(s).$$

Since the function contains  $e^{-2t}$ , the shift is

$$a = -2 \implies F(s - a) = F(s + 2).$$

Therefore

$$\mathcal{L}[te^{-2t}] = F(s + 2) = \boxed{\frac{1}{(s + 2)^2}}.$$

## 2.2 Using the Shift Property in Reverse

To solve differential equations, we use the shift property backward:

$$\mathcal{L}^{-1}[F(s - a)] = e^{at}f(t).$$

First, recognize the base function  $F(s)$ . Then identify the shift amount  $a$ . Then apply the formula by substitution.

### EXAMPLE 3

Compute the inverse Laplace transform of

$$X(s) = \frac{s - 1}{(s - 1)^2 + 9}.$$

#### Solution.

The base transform function and its inverse are

$$F(s) = \frac{s}{s^2 + 9}, \quad f(t) = \cos 3t.$$

Next, we observe by comparison that the original function can be written as

$$F(s - 1) = \frac{s - 1}{(s - 1)^2 + 9}.$$

Thus, if  $F(s - a) = F(s - 1)$  then the shift is  $a = 1$ . Finally, we apply the formula and substitute

$$\mathcal{L}^{-1}\left[\frac{s - 1}{(s - 1)^2 + 9}\right] = e^{at}f(t) = \boxed{e^t \cos 3t}.$$

In Example 3, the numerator and denominator contain the same shift  $s - 1$ . When they do not match, we must rewrite the numerator before using the table.

### EXAMPLE 4

Compute the inverse Laplace transform of

$$X(s) = \frac{s}{(s - 1)^2 + 1}$$

#### Solution.

First, compare the function to formulas in the table to identify the base transform. We

see that this resembles the cosine transformation, and so

$$F(s) = \frac{s}{s^2 + 1}, \quad f(t) = \cos t.$$

The denominator suggests a shift  $a = 1$ , but the shifted cosine formula requires  $s - 1$  in the numerator. Write  $s = (s - 1) + 1$  and split the function over addition:

$$\begin{aligned} \frac{s}{(s-1)^2 + 1} &= \frac{(s-1) + 1}{(s-1)^2 + 1} \\ &= \frac{s-1}{(s-1)^2 + 1} + \frac{1}{(s-1)^2 + 1}. \end{aligned}$$

Now apply the inverse transform to each term:

$$\begin{aligned} \mathcal{L}^{-1} \left[ \frac{s}{(s-1)^2 + 1} \right] &= \underbrace{\mathcal{L}^{-1} \left[ \frac{s-1}{(s-1)^2 + 1} \right]}_{\text{shifted cosine}} + \underbrace{\mathcal{L}^{-1} \left[ \frac{1}{(s-1)^2 + 1} \right]}_{\text{shifted sine}} \\ &= e^t \cos t + e^t \sin t. \end{aligned}$$

Therefore,

$$x(t) = e^t (\cos t + \sin t).$$

**Key idea:** match the numerator to the shift in the denominator. Any leftover constant produces a sine term.

### 3. Solving IVPs

#### 3.1 Procedure

Let's review the general procedure for solving IVPs:

1. Transform the entire DE.
2. Solve for  $X$ .
3. Apply the inverse transform.

The third step is most difficult out of the three. The third step may require a combination of

- partial fraction decomposition,
- shift formula, or
- completing the square.

Here is an example of an IVP with repeated roots.

#### EXAMPLE 5

Use Laplace transforms to solve

$$x'' + 4x' + 4x = 0, \quad x(0) = 1, \quad x'(0) = 0.$$

**Solution.**

**Step 1: Transform the DE.**

$$\begin{aligned} \mathcal{L}[x''] + 4\mathcal{L}[x'] + 4\mathcal{L}[x] &= 0, \\ (s^2 X - sx(0) - x'(0)) + 4(sX - x(0)) + 4X &= 0, \\ (s^2 X - s) + 4sX - 4 + 4X &= 0. \end{aligned}$$

**Step 2: Solve for  $X(s)$ .**

$$(s^2 + 4s + 4)X = s + 4 \implies X(s) = \frac{s + 4}{s^2 + 4s + 4}.$$

**Step 3: Apply the inverse transform.** The denominator factors as  $(s + 2)^2$ , so

$$X(s) = \frac{s + 4}{(s + 2)^2}.$$

We must deal with the  $s$  in the numerator. One option is to partial fraction decomposition

$$X(s) = \frac{s + 4}{(s + 2)^2} = \frac{A}{s + 2} + \frac{B}{(s + 2)^2},$$

but there is a simpler approach. You can add and subtract 2 in the numerator

$$X(s) = \frac{s + 4}{(s + 2)^2} = \frac{s + 4 + 2 - 2}{(s + 2)^2} = \frac{(s + 2) + 2}{(s + 2)^2} = \frac{1}{s + 2} + \frac{2}{(s + 2)^2}$$

The base function in  $\frac{2}{(s+2)^2}$  is  $F(t) = \frac{1}{s^2}$  which has the inverse  $f(t) = t$ . The shift is  $a = -2$ . Therefore, the inverse of the entire function is

$$x(t) = \mathcal{L}^{-1} \left[ \frac{1}{s + 2} + \frac{1}{(s + 2)^2} \right] = \boxed{e^{-2t} + te^{-2t}}.$$

We now consider an equation with complex roots. Its solution should contain an exponential multiplied by sine and cosine.

#### EXAMPLE 6

Use Laplace transforms to solve

$$x'' + 2x' + 5x = 0, \quad x(0) = 1, \quad x'(0) = 0.$$

**Solution.**

**Step 1: Transform the DE.**

$$\begin{aligned} \mathcal{L}[x''] + 2\mathcal{L}[x'] + 5\mathcal{L}[x] &= 0, \\ (s^2X - sx(0) - x'(0)) + 2(sX - x(0)) + 5X &= 0, \\ (s^2X - s) + 2(sX - 1) + 5X &= 0. \end{aligned}$$

**Step 2: Solve for  $X(s)$ .**

$$(s^2 + 2s + 5)X = s + 2 \implies X(s) = \frac{s + 2}{s^2 + 2s + 5}.$$

**Step 3: Apply the inverse transform.** Even though  $X$  is a rational function, partial fraction decomposition will not be effective here. The denominator  $s^2 + 2s + 5$  is *irreducible*.

We want the denominator in the form  $(s - a)^2 + \beta^2$ . Since  $(s + 1)^2 = s^2 + 2s + 1$ , we add and subtract 1:

$$\begin{aligned} s^2 + 2s + 5 &= (s^2 + 2s + 1) + 5 - 1 \\ &= (s + 1)^2 + 4. \end{aligned}$$

This technique is called **completing the square**. Now

$$X(s) = \frac{s+2}{(s+1)^2+4}.$$

The denominator reveals the shift  $a = -1$  and frequency  $\beta = 2$ . To match the shifted cosine formula, write the numerator as  $s+2 = (s+1) + 1$ :

$$\begin{aligned} X(s) &= \frac{s+1}{(s+1)^2+4} + \frac{1}{(s+1)^2+4} \\ &= \frac{s+1}{(s+1)^2+2^2} + \frac{1}{2} \frac{2}{(s+1)^2+2^2}. \end{aligned}$$

**Step 4: Apply the inverse transform.** The first term gives a shifted cosine, and the second gives a shifted sine:

$$x(t) = e^{-t} \cos 2t + \frac{1}{2} e^{-t} \sin 2t.$$

Therefore,

$$x(t) = e^{-t} \left( \cos 2t + \frac{1}{2} \sin 2t \right).$$

The characteristic roots are  $\lambda = -1 \pm 2i$ , so this agrees with the expected form: an exponential decay multiplied by a wave.

### 3.2 Completing the Square

In Example 6, we rewrote the denominator to reveal the shift. In general, we complete the square by taking half the coefficient of  $s$ , squaring it, and adding and subtracting that quantity:

$$\begin{aligned} s^2 + bs + c &= \left( s^2 + bs + \frac{b^2}{4} \right) + c - \frac{b^2}{4} \\ &= \left( s + \frac{b}{2} \right)^2 + c - \frac{b^2}{4}. \end{aligned}$$

For an irreducible quadratic,  $c - b^2/4 > 0$ . This gives the form  $(s - a)^2 + \beta^2$  needed for the shifted sine and cosine formulas, where

$$a = -\frac{b}{2}, \quad \beta = \sqrt{c - \frac{b^2}{4}}.$$

If the coefficient of  $s^2$  is not 1, factor it out before completing the square.

#### EXAMPLE 7

Find the inverse transform by completing the square

$$\mathcal{L}^{-1} \left[ \frac{7}{s^2 + 2s + 50} \right].$$

**Solution.**

The denominator is an irreducible quadratic. Complete the square to reveal the shift:

$$s^2 + 2s + 50 = (s + 1)^2 + 49.$$

Thus,

$$X(s) = \frac{7}{(s + 1)^2 + 7^2}.$$

The base transform is

$$F(s) = \frac{7}{s^2 + 7^2}, \quad f(t) = \sin(7t).$$

Since  $X(s) = F(s + 1) = F(s - (-1))$ , the shift is  $a = -1$ . Therefore,

$$x(t) = \mathcal{L}^{-1}[X(s)] = e^{-t} \sin(7t).$$

## 4. Choosing the Right Formula

We can now solve homogeneous second-order IVPs with all four root types from our opening checklist. The roots of the characteristic equation

$$a\lambda^2 + b\lambda + c = 0$$

help us anticipate the form of  $X(s)$  and choose the inverse transform.

### 4.1 Distinct Real Roots

For roots  $\lambda_1 \neq \lambda_2$ , your transformed solution should be expressed in the form

$$X(s) = \frac{A}{s - \lambda_1} + \frac{B}{s - \lambda_2}.$$

This will require **partial fraction decomposition**. Each fraction produces an exponential:

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}.$$

The shift formula is not required for this case.

### 4.2 Repeated Real Root

For a repeated root  $\lambda$ , your transformed solution should be expressed in the form

$$X(s) = \frac{A}{s - \lambda} + \frac{B}{(s - \lambda)^2}.$$

To get your solution in this form you may need **partial fraction decomposition**. Then you use the **shift formula** on the second term to get the solution

$$x(t) = Ae^{\lambda t} + Bte^{\lambda t}.$$

### 4.3 Complex Roots

For roots  $\alpha \pm i\beta$ , you should aim to write your transformed solution in the form

$$X(s) = A \frac{s - \alpha}{(s - \alpha)^2 + \beta^2} + B \frac{\beta}{(s - \alpha)^2 + \beta^2}.$$

You will need to **complete the square**. Then apply the **shift formula** to obtain:

$$x(t) = e^{\alpha t} (A \cos \beta t + B \sin \beta t).$$

Purely imaginary roots are the special case  $\alpha = 0$ , so no exponential shift is needed.