

3.2 - Solving IVPs Using Tables, and Partial Fractions

(Section 3.2.1)

1. Inverse Transforms

1.1 Goal

The goal of this lecture is to learn how to solve the initial value problem

$$ax'' + bx' + cx = f(t), \quad x(0) = x_0, \quad x'(0) = v_0$$

using Laplace transforms. Here is the general idea of how this works:

1. Transform the entire differential equation into one containing no derivatives

$$\underbrace{\mathcal{L}[ax'' + bx' + cx = f]}_{\text{DE}} \rightarrow \underbrace{F(X) = B}_{\text{algebraic}}.$$

2. Solve for X in the transform domain

$$X = F^{-1}(B).$$

3. Then undo the transform to bring the solution back into the time domain

$$x(t) = \mathcal{L}^{-1}[X].$$

Let's briefly discuss how to perform the last step.

1.2 Inverse Laplace

Throughout the lecture notes, we will use x and t for the time domain, and X and s for the transform domain. The notation for the transform of $x(t)$ is denoted

$$\mathcal{L}[x(t)] = X(s).$$

We define the **inverse Laplace** to be the operator $\mathcal{L}^{-1}[\cdot]$ satisfying

$$\mathcal{L}^{-1}[X(s)] = x(t).$$

There is a formula to find the inverse Laplace transform, and it looks like this:

$$\mathcal{L}^{-1}[X(s)] = \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} X(s) ds.$$

However, we will never make direct use of the definition. This formula is linear in X , so the inverse Laplace transform is linear.

$$\mathcal{L}^{-1}[aX + bY] = a\mathcal{L}^{-1}[X] + b\mathcal{L}^{-1}[Y].$$

Linearity and the table of transforms are all we need to solve differential equations.

1.3 The Table

So far, we have learned 5 transforms. They are listed in the table below. You are not responsible for memorizing any of these formulas. We will use the formulas in reverse. Each formula

is numbered.

	$x(t)$	$X(s)$
1.	1	$\frac{1}{s}$
2.	t	$\frac{1}{s^2}$
3.	e^{at}	$\frac{1}{s-a}$
4.	$\cos(at)$	$\frac{s}{s^2+a^2}$
5.	$\sin(at)$	$\frac{a}{s^2+a^2}$

Let's practice finding a few inverse transforms. You should get in the habit of citing which formulas you are referencing.

EXAMPLE 1

Compute the inverse Laplace transforms of the following functions:

1. $X(s) = \frac{2}{s}$
2. $X(s) = \frac{1}{s-1} + \frac{9}{s^2}$
3. $X(s) = \frac{17}{s^2+4}$

Solution.

1. Using Formula (1), the inverse is

$$\mathcal{L}^{-1} \left[\frac{2}{s} \right] = 2 \mathcal{L}^{-1} \left[\frac{1}{s} \right] = 2(1) = \boxed{2}.$$

2. We may split the operator over addition. Then we use Formulas (2) and (3):

$$\mathcal{L}^{-1} \left[\frac{1}{s-1} + \frac{9}{s^2} \right] = \mathcal{L}^{-1} \left[\frac{1}{s-1} \right] + 9 \mathcal{L}^{-1} \left[\frac{1}{s^2} \right] = \boxed{e^t + 9t}.$$

3. Notice that $X(s) = \frac{17}{s^2+4}$ resembles the transform of $\sin(2t)$ but the numerator is different. Using Formula (5) and some manipulation, we get

$$\mathcal{L}^{-1} \left[\frac{17}{s^2+4} \right] = 17 \cdot \frac{2}{2} \mathcal{L}^{-1} \left[\frac{1}{s^2+4} \right] = \frac{17}{2} \mathcal{L}^{-1} \left[\frac{2}{s^2+4} \right] = \boxed{\frac{17}{2} \sin 2t}.$$

2. Solving DEs with Laplace Transforms

2.1 First Derivative Formula

In order to use the Laplace transform to solve DEs, we must first establish the transform of the derivative. Suppose $X(s) = \mathcal{L}[x(t)]$. Then

$$\boxed{\mathcal{L}[x'(t)] = sX(s) - x(0)}$$

Notice that the initial value $x(0)$ naturally results from the formula. Let's do a first-order IVP example to demonstrate how to use this formula.

EXAMPLE 2

Use the Laplace transform to solve the IVP

$$x' + 3x = 0, \quad x(0) = 2.$$

Solution.

Step 1: Take Laplace transforms of both sides of the equation.

$$\begin{aligned}\mathcal{L}[x' + 3x] &= \mathcal{L}[0] \\ \mathcal{L}[x'] + 3\mathcal{L}[x] &= 0\end{aligned}$$

Substitute the derivative transform.

$$\begin{aligned}(sX - x(0)) + 3X &= 0 \\ (sX - 2) + 3X &= 0\end{aligned}$$

Step 2: Solve for X algebraically.

$$(sX - 2) + 3X = 0 \implies (s + 3)X = 2 \implies X = \frac{2}{s + 3}$$

Step 3: Apply the inverse transform

$$x(t) = \mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}\left[\frac{2}{s + 3}\right] = 2\mathcal{L}^{-1}\left[\frac{1}{s + 3}\right] = 2(e^{-3t}).$$

We have found the solution

$$x(t) = 2e^{-3t}.$$

2.2 Proof of the Formula

We will prove that

$$\mathcal{L}[x'(t)] = sX(s) - x(0).$$

Proof. We begin by assuming this product vanishes $x(t)e^{-st} \rightarrow 0$ as $t \rightarrow \infty$. For integration by parts, choose

$$u = e^{-st}, \quad dv = x'(t) dt, \quad du = -se^{-st} dt, \quad v = x(t).$$

Then

$$\begin{aligned}\mathcal{L}[x'(t)] &= \int_0^\infty x'(t)e^{-st} dt && \text{(Definition)} \\ &= [x(t)e^{-st}]_0^\infty + s \int_0^\infty x(t)e^{-st} dt && \text{(Integration by parts)} \\ &= 0 - x(0) + sX(s) && \text{(Boundary term vanishes at } \infty) \\ &= sX(s) - x(0).\end{aligned}$$

□

2.3 Second Derivative Formula

To solve a 2nd order DE we need the following formula:

$$\mathcal{L}[x''(t)] = s^2X(s) - sx(0) - x'(0)$$

Notice how the two initial values $x(0)$ and $x'(0)$ arise. You do not have to memorize this equation. It is included in the table of transforms which is provided on the exam.

2.4 Proof of the Second Derivative Formula

The proof of the second-order formula is just a repeated application of the first-order formula.

Proof.

$$\begin{aligned}\mathcal{L}[x''(t)] &= s\mathcal{L}[x'(t)] - x'(0) \\ &= s(sX(s) - x(0)) - x'(0) \\ &= s^2X(s) - sx(0) - x'(0).\end{aligned}$$

□

We can begin solving simple second-order DEs.

EXAMPLE 3

Solve

$$x'' + 4x = 0, \quad x(0) = 1, \quad x'(0) = 0.$$

Solution.

Step 1: Take Laplace transforms

$$\mathcal{L}[x''] + 4\mathcal{L}[x] = 0$$

Substitute derivative transforms. Note that $\mathcal{L}[x]$ is simply X .

$$\begin{aligned}(s^2X - sx(0) - x'(0)) + 4X &= 0 \\ (s^2X - s) + 4X &= 0\end{aligned}$$

Step 2: Solve for $X(s)$

$$\begin{aligned}(s^2 + 4)X &= s \\ X(s) &= \frac{s}{s^2 + 4}\end{aligned}$$

Step 3: Apply the inverse transform. Directly using the table we find the inverse transform

$$x(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2 + 4}\right] = \cos 2t.$$

There is a small flaw in this methodology. Suppose you successfully perform Steps 1 and 2, and you obtain a formula for $X(s)$. What happens if $X(s)$ does not resemble any of the functions in the Laplace Table? The remaining challenge is recognizing the inverse transform. The next example will address such a case.

EXAMPLE 4

Attempt to solve the IVP

$$x'' + 3x' + 2x = 0, \quad x(0) = 1, \quad x'(0) = 0$$

using Laplace transforms.

Solution.

Step 1: Take Laplace transforms

$$\mathcal{L}[x''] + 3\mathcal{L}[x'] + 2\mathcal{L}[x] = 0.$$

Substitute derivative transforms.

$$(s^2 X - s) + 3(sX - 1) + 2X = 0$$

Step 2: Solve for X

$$(s^2 + 3s + 2)X = s + 3$$

$$X = \frac{s + 3}{s^2 + 3s + 2}$$

Step 3: Use the table to find the inverse.

$$X = \frac{s + 3}{s^2 + 3s + 2}.$$

We are stuck here. There is no formula in the table for this particular function.

But of course, we know that there is a solution to $x'' + 3x' + 2x = 0$. In fact, we know the general solution should be

$$x(t) = c_1 e^{-t} + c_2 e^{-2t}.$$

The Laplace transforms of the exponential functions are hidden in $X(s)$.

3. Partial Fraction Decomposition

In many problems, the function $X(s)$ will not match a formula in the Laplace transform table. When this happens, we may be able to rewrite the function as a sum of simpler fractions using *partial fraction decomposition*.

3.1 Definitions

A **rational function** is a ratio of two polynomials $P(s)/Q(s)$, where Q is not the zero polynomial. In this lecture, we consider **proper rational functions**, meaning the degree of P is less than the degree of Q . **Partial fraction decomposition** rewrites a rational function as a sum of simpler rational functions. The form of the decomposition depends on the factors of the denominator. There are two common types of factors:

- **Linear factors** such as $(s + 1)$ or $(s - 3)$
- **Irreducible quadratic factors** such as $(s^2 + 4)$

3.2 Two Linear Factors in the Denominator

Suppose the denominator factors as

$$(s + a)(s + b)$$

where a and b are distinct. Then we write

$$\frac{P(s)}{(s + a)(s + b)} = \frac{A}{s + a} + \frac{B}{s + b}.$$

The constants A and B are chosen so that the two sides are equal. For example, the rational function can be factored like so:

$$X = \frac{s + 3}{s^2 + 3s + 2} = \frac{s + 3}{(s + 1)(s + 2)}.$$

Assume

$$\frac{s+3}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}.$$

Multiply both sides by $(s+1)(s+2)$.

$$s+3 = A(s+2) + B(s+1).$$

Solve for A and B .

$$A = 2, \quad B = -1.$$

Thus

$$X = \frac{2}{s+1} - \frac{1}{s+2}.$$

EXAMPLE 4 (cont.)

Complete the problem

$$x'' + 3x' + 2x = 0, \quad x(0) = 1, \quad x'(0) = 0,$$

where we found

$$X = \frac{s+3}{s^2 + 3s + 2}.$$

Solution.

Now we know that we can use PFD to rewrite the function as

$$X = \frac{2}{s+1} - \frac{1}{s+2}.$$

Now each term matches a formula in the transform table. We can find the inverse transform using linearity:

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}[X] \\ &= \mathcal{L}^{-1}\left[\frac{2}{s+1} - \frac{1}{s+2}\right] \\ &= \mathcal{L}^{-1}\left[\frac{2}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] \\ &= 2\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] \\ &= 2e^{-t} - e^{-2t} \end{aligned}$$

Therefore, we have found the solution to the IVP

$$x(t) = 2e^{-t} - e^{-2t}.$$

3.3 Anticipating the Solution

It is helpful to analyze the differential equations before attempting to solve them with Laplace transforms. Suppose your goal is to solve the second-order DE

$$ax'' + bx' + cx = 0$$

and you notice that the eigenvalues are real and distinct. The characteristic equation factors cleanly:

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0.$$

That means the general solution should have the form

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

Knowing this ahead of time, we should expect a certain form for $X(s)$. We should anticipate a form

$$X(s) = \frac{c_1}{s - \lambda_1} + \frac{c_2}{s - \lambda_2},$$

which, when unfactorized, may appear as

$$X(s) = \frac{As + B}{(s - \lambda_1)(s - \lambda_2)}.$$

Think about the nonhomogeneous problem

$$x'' + 9x = 4e^t.$$

What kind of $X(s)$ can we expect from this equation based on what we know about the form of the solution?

3.4 Irreducible Quadratic

When it comes to rational functions, you must be able to identify reducible quadratics versus irreducible quadratics. For example, this quadratic is reducible

$$\underbrace{\frac{3s + 1}{(s^2 - 4)(s - 1)}}_{\text{quadratic}} = \underbrace{\frac{3s + 1}{(s - 2)(s + 2)(s - 1)}}_{\text{linears}}.$$

Sometimes the denominator contains a quadratic that cannot be factored. For example,

$$\frac{2s + 1}{(s^2 + 9)(s - 1)}.$$

In this case, the decomposition must contain a linear term in the numerator

$$\frac{2s + 1}{(s^2 + 9)(s - 1)} = \frac{As + B}{s^2 + 9} + \frac{C}{s - 1}.$$

EXAMPLE 5

Find the inverse Laplace transform of the function

$$X(s) = \frac{5s^2 + 5}{(s^2 + 9)(s - 1)} = \frac{As + B}{s^2 + 9} + \frac{C}{s - 1}.$$

Solution.

Step 1: Multiply both sides by $(s^2 + 9)(s - 1)$:

$$\begin{aligned} 5s^2 + 5 &= (As + B)(s - 1) + C(s^2 + 9) \\ 5s^2 + 5 &= As^2 - As + Bs - B + Cs^2 + 9C. \end{aligned}$$

Step 2: Group like powers of s and derive a system of equations:

$$(5)s^2 + (0)s + (5) = (A + C)s^2 + (-A + B)s + (9C - B).$$

System:

$$\begin{cases} A + C = 5 \\ -A + B = 0 \\ 9C - B = 5. \end{cases}$$

Step 3: Solve the system:

$$C = 1, \quad A = 4, \quad B = 4.$$

Therefore, the decomposition is

$$X(s) = \frac{4s + 4}{s^2 + 9} + \frac{1}{s - 1}.$$

Step 4: Apply the inverse transform using linearity and the table

$$X(s) = \frac{4s + 4}{s^2 + 9} + \frac{1}{s - 1} = 4\frac{s}{s^2 + 9} + 4\frac{1}{s^2 + 9} + \frac{1}{s - 1}.$$

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}[X] \\ &= \mathcal{L}^{-1}\left[4\frac{s}{s^2 + 9} + 4\frac{1}{s^2 + 9} + \frac{1}{s - 1}\right] \\ &= 4\mathcal{L}^{-1}\left[\frac{s}{s^2 + 9}\right] + \frac{4}{3}\mathcal{L}^{-1}\left[\frac{3}{s^2 + 9}\right] + \mathcal{L}^{-1}\left[\frac{1}{s - 1}\right] \\ &= 4\cos(3t) + \frac{4}{3}\sin(3t) + e^t. \end{aligned}$$