

2.7 – Forced Springs: Beats and Resonance

(Section 2.3.2)

1. Forced Vibrations with Damping

1.1 Spring-Mass System

A spring-mass system driven by an external force satisfies

$$\underbrace{mx''}_{\text{inertia}} + \underbrace{\gamma x'}_{\text{damping}} + \underbrace{kx}_{\text{restoring}} = \underbrace{f(t)}_{\text{external}}.$$

Each term represents a force. We focus on sinusoidal forcing,

$$f(t) = F_0 \cos(\omega t),$$

where F_0 is the force amplitude in newtons and ω is the **forcing frequency**. A motorized piston moving the spring's attachment point back and forth can produce this type of forcing.

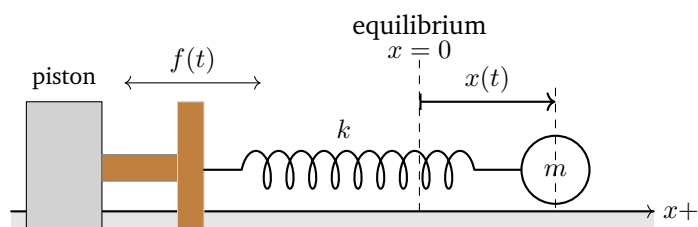


Figure 1: A motorized piston drives the spring-mass system. Displacement is measured from the equilibrium position with the piston at its reference position.

1.2 Levels of Specificity

Recall that nonhomogeneous solutions have the form $x = x_h + x_p$. Depending on the question, we may need only the form of these pieces or their exact coefficients. For example, consider

$$x'' + 2x' + 5x = 25t, \quad x(0) = 0, \quad x'(0) = 5.$$

1. **General form:** find x_h and choose a trial for x_p , leaving its coefficients undetermined:

$$x = e^{-t}(c_1 \cos 2t + c_2 \sin 2t) + At + B.$$

2. **General solution:** determine A and B by substitution:

$$x = e^{-t}(c_1 \cos 2t + c_2 \sin 2t) + 5t - 2.$$

3. **Solution to the IVP:** use the initial conditions to determine c_1 and c_2 :

$$x = e^{-t}(2 \cos 2t + \sin 2t) + 5t - 2.$$

In this lecture, we mainly use the **general form** to predict behavior without calculating every coefficient.

EXAMPLE 1

A spring-mass system has mass $m = 1$ kg, damping coefficient $\gamma = 3$ kg/s, and spring constant $k = 2$ N/m. An external force $f(t) = 10 \cos t$ N acts on the mass. Initially, the

mass is displaced 30 cm to the right and given a rightward velocity of 1 m/s. Formulate the IVP and find the general form of the solution. Do not determine the coefficients.

Solution.

Substituting into $mx'' + \gamma x' + kx = f(t)$ gives $x'' + 3x' + 2x = 10 \cos t$. We treat displacement to the right of the equilibrium as positive. So the displacement 30 cm = 0.3 m is written $x(0) = 0.3$. Our model is

$$\begin{cases} x'' + 3x' + 2x = 10 \cos t, \\ x(0) = 0.3, \\ x'(0) = 1. \end{cases}$$

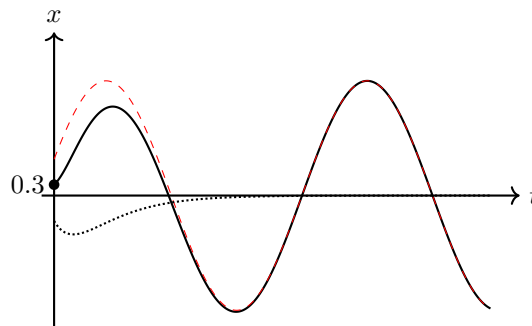
The characteristic equation is

$$\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0,$$

so $x_h(t) = c_1 e^{-t} + c_2 e^{-2t}$. For the forcing term, choose $x_p(t) = A \cos t + B \sin t$. There is no duplication, so the general form is

$$x(t) = \underbrace{c_1 e^{-t} + c_2 e^{-2t}}_{\text{decays}} + \underbrace{A \cos t + B \sin t}_{\text{persists}}.$$

Without finding the coefficients, we can already see that the exponential terms vanish and the sinusoidal part remains. Although the system is **overdamped**, periodic forcing keeps it oscillating. For illustration, the graph shows the solution after the coefficients have been determined.



Solid: full solution. Dashed: persistent part. Dotted: decaying part.

1.3 Transient vs. Steady-State Solution

The two behaviors in Example 1 have names. For a model with forced vibrations and damping, the solution has the general form

$$x(t) = \underbrace{x_h(t)}_{\text{transient}} + \underbrace{A \cos(\omega t) + B \sin(\omega t)}_{\text{steady-state}}.$$

The **transient solution** x_h is a short-lived function which decays to zero due to the friction in the system. The sinusoidal particular solution is the **steady-state solution**. The transient solution captures the contribution from the initial conditions, while the steady-state solution captures the response to the forced vibration.

We will always obtain transient and steady state solutions from the model

$$mx'' + \gamma x' + kx = F_0 \cos(\omega t),$$

if $\gamma \neq 0$. If $\gamma = 0$, then we may obtain more interesting solutions.

2. Forced Vibrations without Damping

2.1 Natural and Forcing Frequencies

If we assume that friction is negligibly small ($\gamma = 0$), then our model for a spring driven by a sinusoidal force satisfies

$$mx'' + kx = F_0 \cos(\omega t).$$

Dividing by m gives

$$x'' + \omega_0^2 x = F \cos(\omega t), \quad \omega_0 = \sqrt{\frac{k}{m}}, \quad F = \frac{F_0}{m}.$$

There are two angular frequencies to distinguish:

- ω_0 : the **natural frequency**, determined by the mass and spring constant.
- ω : the **forcing frequency**, determined by the external source.

The homogeneous solution (solution to $x'' + \omega_0^2 x = 0$) is

$$x_h(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t).$$

In the absence of an external force, the spring oscillates according to the natural frequency ω_0 . Since we assume there is no friction, the homogeneous solution does not vanish.

2.2 Three Cases

The relationship between the two frequencies determines the behavior:

1. **Far apart**: a fast wave combines with a slow wave.
2. **Close but unequal**: waves of comparable amplitude produce **beats**.
3. **Equal**: nonzero sinusoidal forcing produces **resonance**, with unbounded growth.

2.3 Distinct Frequencies

If $\omega \neq \omega_0$, the trial function

$$x_p(t) = A \cos(\omega t) + B \sin(\omega t)$$

does not overlap with x_h . Thus, the general form is

$$x(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + A \cos(\omega t) + B \sin(\omega t).$$

We can already see that the solution is bounded: it is a sum of waves with constant coefficients.

EXAMPLE 2

Find the general form of the solution to

$$x'' + 100x = 99 \cos t.$$

Solution.

The characteristic roots are $\lambda = \pm 10i$, so $\omega_0 = 10$. The forcing frequency is $\omega = 1$. There is no duplication, giving

$$x(t) = c_1 \cos 10t + c_2 \sin 10t + A \cos t + B \sin t.$$

The motion combines waves with frequencies 10 and 1.

2.4 Interference

Adding waves produces **interference**: they reinforce one another when they have the same sign and partially or completely cancel when they have opposite signs. To illustrate, take the solution from Example 2 with $c_1 = A = 1$ and $c_2 = B = 0$:

$$x(t) = \underbrace{\cos 10t}_{\text{fast wave}} + \underbrace{\cos t}_{\text{slow wave}}.$$

These coefficients satisfy the DE; we use them here to focus on the shape. The fast wave oscillates around zero. Adding $\cos t$ moves its centerline to the dashed curve below. There are ten fast cycles during one slow cycle.

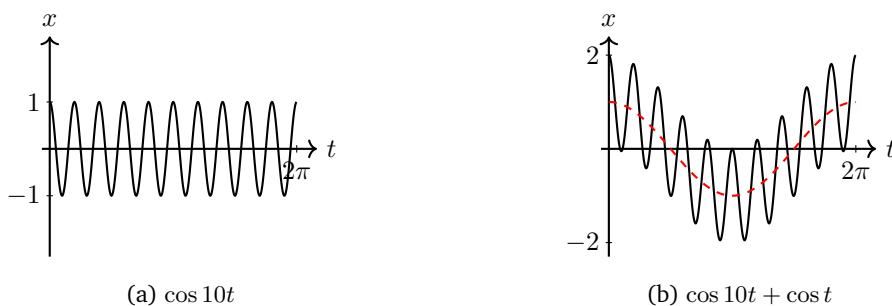


Figure 2: Adding $\cos t$ shifts the fast oscillation onto a moving centerline (dashed).

Next, we consider close but unequal frequencies, which can produce **beats**.

3. Beats

3.1 Close Frequencies

Desmos experiment. On Desmos, plot the function $y = \cos(ax) + \cos(bx)$ with sliders for a and b . Fix $b = 30$ and increase a from 1 to 29. Observe how the interference pattern changes as the frequency a approaches b . The pattern that emerges is called a *beat*.

A **beat** is a pattern of repeated growth and decline in amplitude produced by waves with close but unequal frequencies. Consider the sum

$$x(t) = \cos(\omega t) + \cos(\omega_0 t), \quad \omega \approx \omega_0, \quad \omega \neq \omega_0.$$

The closer the frequencies, the more slowly the amplitude changes. There is no exact cutoff for “close.” For our sketches, a useful guideline is

$$\frac{\text{smaller frequency}}{\text{larger frequency}} \geq 0.8,$$

with the frequencies unequal. If the frequencies are within 80% of each other, it is fair to call it a beat.

3.2 Envelope and Carrier

There is a useful trig identity for sums of waves

$$\cos a + \cos b = 2 \cos \left(\frac{a+b}{2} \right) \cos \left(\frac{a-b}{2} \right).$$

Applying this to the beat function yields

$$\begin{aligned} x(t) &= \cos(\omega t) + \cos(\omega_0 t) \\ &= \underbrace{2 \cos(\omega_e t)}_{\text{slow factor}} \underbrace{\cos(\omega_c t)}_{\text{fast factor}}, \end{aligned}$$

where

$$\omega_e = \frac{|\omega - \omega_0|}{2}, \quad \text{and} \quad \omega_c = \frac{\omega + \omega_0}{2}.$$

We call ω_e the **envelope frequency** and ω_c the **carrier frequency**. Close frequencies give $\omega_e \ll \omega_c$. The reason why ω_e is called the "envelope" frequency is because the curves $\pm 2 \cos(\omega_e t)$ form the outer layer for the beat function.

3.3 Sketching Beats

To sketch a beat:

1. Rewrite the sum or difference as a product.
2. Draw the positive and negative slow factors as dashed envelope curves.
3. Find the envelope and carrier periods.
4. Draw the fast oscillation between the envelopes.

EXAMPLE 3

Rewrite $x(t) = \cos 28t + \cos 30t$ as a product and sketch it, indicating the envelope and its period.

Solution.

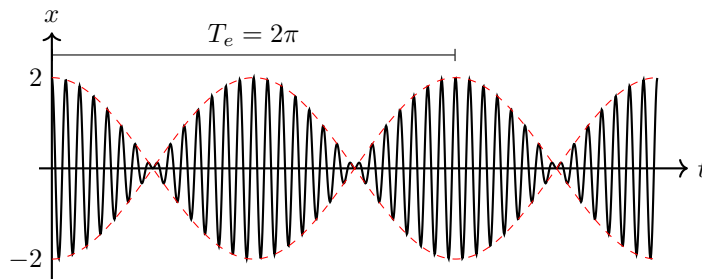
The sum-to-product identity gives

$$x(t) = 2 \cos(29t) \cos t.$$

The envelope curves and periods are

$$x = \pm 2 \cos t, \quad T_e = 2\pi, \quad T_c = \frac{2\pi}{29}.$$

Thus, the carrier completes 29 cycles during one period of the slow cosine. Begin at $x(0) = 2$ and draw the fast oscillation between the dashed curves.



Dashed: envelope curves $\pm 2 \cos t$. There are 29 carrier cycles over the marked interval.

4. Resonance

4.1 Matching Frequencies

Consider an undamped system whose forcing frequency matches its natural frequency:

$$x'' + \omega_0^2 x = F \cos(\omega_0 t), \quad \omega_0 > 0, \quad F \neq 0.$$

The usual trial function duplicates the homogeneous solution, so we multiply it by t :

$$x_p(t) = t(A \cos(\omega_0 t) + B \sin(\omega_0 t)).$$

Thus, even before determining the coefficients, the general form of the solution is

$$x(t) = \underbrace{c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)}_{\text{bounded oscillation}} + \underbrace{t(A \cos(\omega_0 t) + B \sin(\omega_0 t))}_{\text{growing oscillation}}.$$

The factor t produces unbounded growth despite the bounded forcing. This is **resonance** in the undamped model. The bounded homogeneous part cannot cancel this growth.

4.2 Recognizing Resonance

For unbounded resonance, we require no damping. Next, we must sinusoidal forcing term

$$mx'' + kx = F_0 \cos(\omega t) \quad \text{or} \quad mx'' + kx = F_0 \sin(\omega t).$$

Lastly, the innate frequency of the system $\sqrt{\frac{k}{m}}$ must match the forcing frequency

$$\omega = \sqrt{\frac{k}{m}} = \omega_0.$$

These are the key requirements for a forced vibration to exhibit resonance.

EXAMPLE 4

An undamped spring has stiffness $k = 12$ N/m and is driven by the force $f(t) = 10 \sin 6t$ N. What mass produces resonance?

Solution.

The model is

$$mx'' + 12x = 10 \sin 6t.$$

Matching the natural and forcing frequencies gives

$$\sqrt{\frac{12}{m}} = 6 \implies \frac{12}{m} = 36 \implies \boxed{m = \frac{1}{3} \text{ kg.}}$$

We can distinguish resonance, beats, and ordinary interference by comparing the natural and forcing frequencies.

EXAMPLE 5

Classify the expected behavior as **resonance**, **beats**, or **ordinary interference**. For unequal frequencies, assume both wave components are present with comparable amplitudes.

1. $x'' + 4x = \cos t$
2. $x'' + 49x = 2 \cos 50t$
3. $x'' + 64x = \sin 7t$
4. $2x'' + 8x = 4 \sin 2t$

Solution.

First identify $\omega_0 = \sqrt{k/m}$ and the forcing frequency ω . Equal frequencies give resonance. For unequal frequencies, use our 80% guideline to recognize beats:

$$r = \frac{\min(\omega_0, \omega)}{\max(\omega_0, \omega)}.$$

Equation	ω_0	ω	r	Behavior
1	2	1	0.50	Ordinary interference
2	7	50	0.14	Ordinary interference
3	8	7	0.875	Beats
4	2	2	1	Resonance

In equation 4, divide by 2 first:

$$x'' + 4x = 2 \sin 2t.$$

Thus, $\omega_0 = 2 = \omega$, so the response is resonant.

Beats are also an interference pattern; the close frequencies produce a slowly varying amplitude.

5. Connections to Sound and Other Waves

5.1 Sound Waves

At a fixed location, a pure sound tone can be modeled by an oscillating pressure:

$$p(t) = A \cos(\omega t).$$

The frequency ω determines the pitch, while the amplitude A affects the loudness. Here, ω is measured in radians per second; the frequency in hertz is $f = \omega/(2\pi)$.

Suppose two speakers play different tones simultaneously. Each produces its own pressure variation:

$$p_1(t) = A_1 \cos(\omega_1 t), \quad p_2(t) = A_2 \cos(\omega_2 t).$$

At the listener's position, these variations add:

$$p(t) = p_1(t) + p_2(t) = A_1 \cos(\omega_1 t) + A_2 \cos(\omega_2 t).$$

For simplicity, we have chosen zero phase shifts. The resulting pressure is the combined sound that reaches the listener's ear.

This is the same **interference** we studied earlier: the waves reinforce one another when their pressure variations have the same sign and partially or completely cancel when they have opposite signs.

5.2 Hearing Beats

Tuning a piano. A tuning fork produces a reference tone, and striking a piano key produces another tone. When played together, their sound waves add. If the piano is perfectly tuned to

the fork, we hear a steady tone without beats. If it is slightly out of tune, the waves move into and out of step. The sound grows louder and softer: “waa... waa... waa...” These pulses are called **beats**. As the piano’s frequency approaches the fork’s frequency, the beats slow down and eventually disappear. This lets the tuner hear when the note is in tune.

AM radio. A changing amplitude can also carry information. In **amplitude modulation** (AM), a transmitter varies the amplitude of a fast carrier wave according to the speech or music being broadcast:

$$p(t) = (A + s(t)) \cos(\omega_c t), \quad A > |s(t)|.$$

Here, $s(t)$ is the audio signal and $p(t)$ is the electrical signal sent to the antenna. The high-frequency carrier allows efficient transmission, while its changing amplitude carries the sound. Different stations use different carrier frequencies, allowing your radio to select one station and recover its audio.

5.3 Resonance and Its Applications

Pushing a swing. A playground swing has a natural rhythm. If you time your kicks to match that rhythm, each push adds energy and the swing goes higher. This illustrates resonance: a repeated force reinforces the motion when applied at the right frequency. Friction and air resistance keep a real swing from growing without limit.

Shattering a wine glass. Tapping a wine glass makes it ring at one of its natural frequencies. A singer who sustains a matching pitch can drive that vibration through the pressure of the sound. If the sound is sufficiently intense and the glass has little damping, the vibration can become large enough to shatter it. If you’ve seen old cartoons of people breaking glass with their voice, that is an example of resonance.

The Tacoma Narrows Bridge. Structures such as bridges have natural frequencies at which they tend to vibrate. In 1940, wind caused increasingly large twisting motions in the Tacoma Narrows Bridge until it collapsed. This is often presented as resonance: the wind supposedly matched the bridge’s natural frequency.

However, the main mechanism is now believed to be *flutter*! As the bridge twisted, it changed the airflow. The wind then reinforced the twisting, making the motion grow. Unlike our model, the force depended on the bridge’s own motion.

Magnetic resonance imaging (MRI). MRI uses resonance to take pictures inside the body. A strong magnet and carefully tuned radio waves cause hydrogen nuclei in the body to respond and produce signals. The scanner detects these signals and uses them to build images of organs and other tissues. The key idea is familiar: choosing the right frequency makes the system respond.