

2.3 - Characteristic Equation: Repeated Roots

(Section 2.2.2)

1. The Issue with Repeated Roots

Let's first go over what the issue is with repeated roots.

1.1 Characteristic Equation Method

Here are the steps to solve the second order IVP

$$\begin{cases} ax'' + bx' + cx = 0, \\ x(0) = x_0, \\ x'(0) = v_0. \end{cases}$$

1. First write the characteristic equation

$$a\lambda^2 + b\lambda + c = 0.$$

2. Solve for the eigenvalues λ_1 and λ_2 .
3. Write the general solution as the linear combination

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

4. Use the initial conditions to solve for C_1 and C_2 .

Warning: these steps only apply when the two eigenvalue λ_1 and λ_2 are real and distinct.

1.2 Wrong Attempt

Consider the problem

$$\begin{cases} x'' - 4x' + 4x = 0, \\ x(0) = 1, \\ x'(0) = 0. \end{cases}$$

The characteristic equation can be factored cleanly, giving us two eigenvalues $\lambda_1 = \lambda_2 = 2$.

$$\lambda^2 - 4\lambda + 4 = 0 \quad \Rightarrow \quad (\lambda - 2)(\lambda - 2) = 0.$$

If we naively follow the steps, our general solution is

$$x(t) = C_1 e^{2t} + C_2 e^{2t}.$$

The condition $x(0) = 1$ says $C_1 + C_2 = 1$. The condition $x'(0) = 0$ says $2C_1 + 2C_2 = 0$. However, this linear system

$$\begin{cases} C_1 + C_2 = 1 \\ 2C_1 + 2C_2 = 0 \end{cases}$$

has *no solutions*; it's inconsistent. But the existence and uniqueness theorem tells us that this IVP must have a solution. We made a grave mistake somewhere in this calculation.

1.3 What Went Wrong?

Recall that the combination

$$x(t) = C_1 x_1(t) + C_2 x_2(t)$$

is only a general solution if the two functions x_1 and x_2 are *independent*. In the previous example, we only had one eigenvalue $\lambda = 2$ and one solution

$$x_1(t) = e^{2t}.$$

The mistake was that our second solution was the same as our first $x_2 = x_1$. In fact, because C_1 and C_2 are both arbitrary, they just combine as one:

$$C_1 e^{2t} + C_2 e^{2t} = (C_1 + C_2) e^{2t} = \tilde{C} e^{2t}.$$

Adding a second dependent solution adds no contribution to the solution. When you have repeated eigenvalues, you must find a second independent solution.

2. The Proper Method

Now we will explain how to properly solve the DE in the case of repeated eigenvalues.

2.1 General Solution

Suppose the characteristic equation has a repeated real root λ . Then one solution is

$$x_1(t) = e^{\lambda t}.$$

To obtain a second independent solution x_2 , we multiply the solution x_1 by t :

$$x_2(t) = t x_1(t) = t e^{\lambda t}.$$

Thus, the general solution is the combination of x_1 and x_2

$$x(t) = C_1 e^{\lambda t} + C_2 t e^{\lambda t}.$$

This is a rather non-intuitive result. The next section shows the proof of why this works. You will not be tested on the proof.

2.2 Proof (Optional Reading)

We already know the solution $e^{\lambda t}$. To find another, try $x(t) = v(t)e^{\lambda t}$. Differentiating gives

$$\begin{aligned} x' &= (v' + \lambda v) e^{\lambda t}, \\ x'' &= (v'' + 2\lambda v' + \lambda^2 v) e^{\lambda t}. \end{aligned}$$

Substitute into $ax'' + bx' + cx = 0$ and divide by $e^{\lambda t}$:

$$av'' + (2a\lambda + b)v' + (a\lambda^2 + b\lambda + c)v = 0.$$

Since λ is a repeated root,

$$a\lambda^2 + b\lambda + c = 0, \quad \lambda = -\frac{b}{2a} \quad \Rightarrow \quad 2a\lambda + b = 0.$$

Thus, the equation reduces to

$$av'' = 0 \quad \Rightarrow \quad v'' = 0 \quad \Rightarrow \quad v(t) = A + Bt.$$

Therefore,

$$x(t) = (A + Bt)e^{\lambda t} = Ae^{\lambda t} + Bte^{\lambda t}.$$

The factor of t appears naturally when we integrate $v'' = 0$ twice. Now we can return to the previous IVP and solve it correctly.

EXAMPLE 1

Find the solution to

$$\begin{cases} x'' - 4x' + 4x = 0, \\ x(0) = 1, \\ x'(0) = 0. \end{cases}$$

Solution.

Since the eigenvalue is a repeated root $\lambda = 2$, we write the general solution as

$$x(t) = C_1 e^{2t} + C_2 t e^{2t}.$$

The condition $x(0) = 1$ gives

$$C_1 = 1.$$

Therefore,

$$x(t) = e^{2t} + C_2 t e^{2t}.$$

The derivative is

$$x'(t) = (e^{2t} + C_2 t e^{2t})' = 2e^{2t} + C_2(e^{2t} + 2te^{2t}).$$

Therefore, the condition $x'(0) = 0$ yields

$$x'(0) = 2 + C_2(1 + 0) = 0 \implies C_2 = -2.$$

Therefore the solution is

$$x(t) = e^{2t} - 2te^{2t}.$$

This is the unique solution that exists for all $t \in \mathbb{R}$ as guaranteed by our well-posedness theorem.

3. Sketching the Solution

Our goal is to learn how to sketch the repeated-root solution by hand.

3.1 Sketching Strategy

Repeated-root solutions have the form

$$x(t) = (C_1 + C_2 t)e^{\lambda t},$$

which is “linear times exponential.” As in the previous lecture, we use the initial conditions and end behavior to guide our sketch:

1. Plot the initial point $(0, x_0)$ and indicate the initial slope $x'(0) = v_0$.
2. Find the zero by solving $C_1 + C_2 t = 0$, if it exists.
3. Determine the behavior as $t \rightarrow \infty$ and $t \rightarrow -\infty$.
4. Connect these features using the shape of a linear function times an exponential.

To understand this shape, let’s first look at a simple example.

3.2 General Shape

Consider the function

$$x(t) = te^{-t}.$$

This function is the product of a linear function t and an exponential e^{-t} . As t increases, the linear factor grows while the exponential factor decays. Therefore, direct substitution into

the product gives the indeterminate form

$$\lim_{t \rightarrow \infty} t e^{-t} = \infty \cdot 0.$$

You may use L'Hopital's rule to verify that

$$t e^{-t} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Towards the other end, we have

$$t e^{-t} \rightarrow -\infty \quad \text{as } t \rightarrow -\infty.$$

The resulting graph has a characteristic “bump-and-decay” shape. This is a useful prototype for understanding repeated-root solutions.

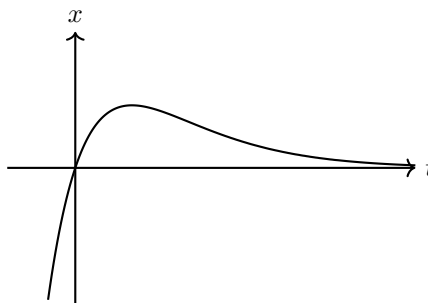


Figure 1: Sketch of $x(t) = t e^{-t}$.

EXAMPLE 2

Sketch the function

$$x(t) = e^{2t} - 2t e^{2t},$$

which satisfies the initial conditions $x(0) = 1$ and $x'(0) = 0$.

Solution.

One of the intercepts is given $(0, 1)$. To find the other intercept, factor out e^{2t} and find the root of the linear factor:

$$\begin{aligned} x(t) &= e^{2t} - 2t e^{2t} \\ &= e^{2t}(1 - 2t) \equiv 0 \\ \implies (1 - 2t) &= 0 \\ \implies t &= 1/2 \end{aligned}$$

Begin by plotting the two points $(0, 1)$ and $(0.5, 0)$. Since $x'(0) = 0$, indicate a horizontal tangent at the point $(0, 1)$. Next, calculate the limits:

$$\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} e^{2t}(1 - 2t) = -\infty,$$

$$\lim_{t \rightarrow -\infty} x(t) = \lim_{t \rightarrow -\infty} e^{2t}(1 - 2t) = 0.$$

Lastly, put all the information together. Draw a bump that peaks at $(0, 1)$ which decays as we move left. To the right, it passes through $(0.5, 0)$ and sinks to $-\infty$.

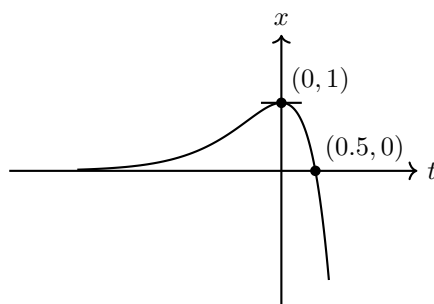


Figure 2: Sketch of $x(t) = (-2t + 1)e^{2t}$

4. From Solution to IVP

So far, we have started with an IVP and found its solution. Can we go in reverse? If we are given a repeated-root solution, can we construct an IVP that produces it?

Yes. In fact, this is how I sometimes construct exam problems with nice, clean solutions. This is mostly a fun exercise, but it is on your homework, so let's see how it works.

4.1 General Steps

Suppose we are given a repeated-root solution

$$x(t) = (C_1 + C_2 t)e^{rt}.$$

We can recover an IVP using the following steps:

1. Write the characteristic equation in factored form:

$$(\lambda - r)^2 = 0.$$

2. Expand:

$$\lambda^2 - 2r\lambda + r^2 = 0.$$

3. Convert the characteristic equation into a differential equation:

$$x'' - 2rx' + r^2x = 0.$$

4. Find the initial position by evaluating $x(0)$.
5. Find the initial velocity by evaluating $x'(0)$.

EXAMPLE 3

Find an IVP whose solution is given by

$$x(t) = 2e^{4t} - te^{4t}.$$

Solution.

The characteristic equation is

$$0 = (\lambda - 4)^2$$

$$0 = \lambda^2 - 8\lambda + 16$$

so the coefficients are $a = 1$, $b = -8$, and $c = 16$. One possible differential equation is

$$x'' - 8x' + 16x = 0.$$

Plugging in $t = 0$ gives

$$x(0) = 2e^0 - (0)e^0 = 2$$

which must be the value of x_0 . Taking the derivative gives us

$$x'(t) = 8e^{4t} - (e^{4t} + 4te^{4t}).$$

Plugging in $t = 0$ into the derivative gives

$$x'(0) = 8e^0 - (e^0 + 4(0)e^0) = 8 - 1 = 7$$

which must be the value of the initial derivative. One IVP would be

$$\begin{cases} x'' - 8x' + 16x = 0, \\ x(0) = 2, \\ x'(0) = 7 \end{cases}.$$

5. Complex Numbers

Since we have time to spare, we can begin discussing the third case: complex eigenvalues.

5.1 Non-real Roots

Consider the DE

$$x'' - 2x' + 5x = 0.$$

The characteristic equation is

$$\lambda^2 - 2\lambda + 5 = 0.$$

We can solve it using the quadratic formula, but we encounter a complication

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm \frac{1}{2}\sqrt{-16}.$$

The issue is that $\sqrt{-16}$ is not a real number. More precisely,

$$\text{there is no } x \in \mathbb{R} \text{ such that } x^2 = -16.$$

The roots do exist, just not in $\mathbb{R}$. We need to introduce a broader class of numbers that allows us to solve equations such as $x^2 + 16 = 0$.

5.2 Imaginary Unit

We define the **imaginary unit** i to be

$$i = \sqrt{-1}.$$

The imaginary unit is not a real number, so $i \notin \mathbb{R}$. It belongs to a larger number system called the **complex numbers**, denoted by $\mathbb{C}$. This new number satisfies

$$i^2 = -1.$$

We can perform ordinary algebra with i . For example,

$$i + 2i = 3i, \quad (2i)(3i) = 6i^2 = -6.$$

More importantly, i allows us to write square roots of negative numbers. For example,

$$\sqrt{-16} = \sqrt{(-1)(16)} = 4i.$$

For our purposes, this is really all we need to know about i . Now we can return to our differential equation.

EXAMPLE 4

Find the *complex* eigenvalues for

$$x'' - 2x' + 5x = 0.$$

Solution.

From the previous subsection, the characteristic equation gives

$$\lambda = 1 \pm \frac{1}{2}\sqrt{-16}.$$

Using the imaginary unit,

$$\sqrt{-16} = 4i,$$

so

$$\lambda = 1 \pm \frac{1}{2}(4i) = 1 \pm 2i.$$

Therefore, the two complex eigenvalues are

$$\boxed{\lambda_1 = 1 + 2i, \quad \lambda_2 = 1 - 2i.}$$

5.3 Standard Form

Any complex number $z \in \mathbb{C}$ can be written in the form

$$z = a + ib,$$

where $a, b \in \mathbb{R}$. This is called the **standard form** of a complex number. The term a is called the **real part**, while b is called the **imaginary part**. For example,

$$z = 3 + 2i$$

has real part 3 and imaginary part 2. Sometimes, it is not so obvious what the real and imaginary parts are. Some algebraic manipulation may be required. Consider the number $z = 5i + 2/i$. To see what the real and imaginary parts are, multiply the 2nd term by i/i :

$$\begin{aligned} z &= 5i + \frac{2}{i} \frac{i}{i} \\ &= 5i + \frac{2i}{i^2} \\ &= 5i - 2i \\ &= 3i \end{aligned}$$

For this complex number, the real part is 0 and the imaginary part is 3.

5.4 Euler's Formula

Complex numbers can appear in the exponential function. In this case, writing the number in standard form requires the help of a theorem.

Theorem 1. If $i = \sqrt{-1}$ then for any $t \in \mathbb{R}$,

$$e^{it} = \cos t + i \sin t.$$

This equation is called **Euler's formula**. Here is a simple use of the formula:

$$e^{i\pi/2} = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = i.$$

The main use of Euler's formula for us is to write complex exponential functions in standard form so that we can identify their real and imaginary parts.

EXAMPLE 6

Write the number

$$z = e^{6+3i}$$

in the standard form $z = a + ib$.

Solution.

Rewrite the exponential as a product and apply Euler's formula

$$\begin{aligned} z &= e^{6+3i} \\ &= e^6 e^{3i} \\ &= e^6 (\cos(3) + i \sin(3)) \\ &= e^6 \cos(3) + i e^6 \sin(3) \end{aligned}$$

The standard form is

$$z = e^6 \cos(3) + i e^6 \sin(3).$$

Thus, the real part is $e^6 \cos(3)$ and the imaginary part is $e^6 \sin(3)$.

5.5 Odd and Even Functions

What would the number

$$z = e^{6-3i}$$

look like when it is written in standard form? Using Euler's formula, we get

$$\begin{aligned} z &= e^{6-3i} \\ &= e^6 e^{-3i} \\ &= e^6 \cos(-3) + i e^6 \sin(-3) \end{aligned}$$

We can clean this up a bit. Recall that $\cos(x)$ is an even function, i.e.

$$\cos(-x) = \cos(x)$$

and that $\sin(x)$ is an odd function, i.e.

$$\sin(-x) = -\sin(x).$$

Therefore

$$z = e^{6-3i} = e^6 \cos(3) - i e^6 \sin(3)$$

5.6 Next Lecture

In Example 4, we found that

$$x'' - 2x' + 5x = 0$$

has eigenvalues $\lambda = 1 \pm 2i$. As before, the eigenvalue $\lambda = 1 + 2i$ gives an exponential solution

$$x_1(t) = e^{(1+2i)t}.$$

However, this solution is complex-valued, while we want real solutions. Next lecture, we will use Euler's formula to obtain the real general solution.