

1.7 – Population Models

(Section 1.5.1)

1. Basic Growth Model

Thomas Malthus was an essayist in the late 1700s who proposed a simple mathematical model for population growth. The model is $P' = rP$, where $P(t)$ is the population at time t and r is a constant.

1.1 Model Assumption

Let $P(t)$ denote the population of a species at time t . The time variable t may be measured in minutes, days, years, etc. Suppose the population grows at a fixed percentage rate r per unit time. Over a small time interval Δt , we approximate the change in population by

$$\underbrace{P(t + \Delta t)}_{\text{future pop.}} \approx \underbrace{P(t)}_{\text{current pop.}} + \underbrace{r \Delta t P(t)}_{\text{growth factor}}$$

With a little rearrangement, we can write this as

$$\frac{P(t + \Delta t) - P(t)}{\Delta t} \approx rP(t).$$

In the limit as $\Delta t \rightarrow 0$, we obtain the continuous **basic growth model**

$$\frac{dP}{dt} = rP.$$

The idea behind the model is simple: if the population is large, there are more individuals capable of reproducing. Therefore, the population growth rate increases with population size.

1.2 Parameter r

The parameter r is called the **intrinsic growth rate**. It represents the net per-capita growth rate of the population and may be interpreted as $r = \text{birth rate} - \text{death rate}$. External factors like migration should not be aggregated into r . The units of r are inverse time. For example,

$$r = 0.15 \text{ month}^{-1}$$

corresponds to a growth rate of 15% per month. We should now be able to translate descriptions of population growth into differential equation models.

EXAMPLE 1

For each species, write the basic growth model that describes its population.

- A population of rabbits grows at a rate of 30% per month.
- A population of bacteria roughly triples per minute.

Solution.

- Let $R(t)$ be the rabbit population after t months. Since the intrinsic growth rate is 30% per month, $r = 30\% = 0.3$. Therefore, the model is

$$\boxed{R' = 0.3R.}$$

- Let $B(t)$ be the bacteria population after t minutes. Population tripling means the same as a 200% growth rate. Therefore, the intrinsic growth rate is $r = 200\% = 2$.

The model is

$$B' = 2B.$$

1.3 An Issue with the Model

Suppose we model a growing population by

$$P' = rP, \quad P(0) = P_0,$$

where $r > 0$. The solution is

$$P(t) = P_0 e^{rt}.$$

This can be a reasonable model over short time spans. However,

$$\lim_{t \rightarrow \infty} P(t) = \lim_{t \rightarrow \infty} P_0 e^{rt} = \infty.$$

In other words, the model predicts that the population will grow *without bound*. The problem is that the basic growth model assumes the same growth rate regardless of population size. In reality, resources such as food, space, and water are limited. As a population becomes large, these limitations begin to slow its growth.

Thus, the basic growth model can be useful over short time spans, but it is usually unrealistic over long time spans. We need a better model that accounts for limited resources. We can create a more realistic model by modifying the basic one.

2. Logistic Growth Model

In the 1830s, Belgian mathematician Verhulst developed a model to describe self-limiting population growth, generalizing the earlier exponential growth model of Malthus by incorporating a sustainable population level known as the carrying capacity.

2.1 Carrying Capacity

Every population should have a soft upper bound due to the limited resources in an ecosystem. Let $K > 0$ represent a sustainable population level. The parameter K is called the **carrying capacity**, and it has the same units as population. We want three things to be true:

- If the population is below K , the population should rise.
- As the population approaches K , its growth should slow down.
- If the population is above K , the population should fall.

2.2 The Model

To enforce the three properties of the carrying capacity K , we modify the basic growth model as follows:

$$P' = rP \quad \rightarrow \quad P' = rP \left(1 - \frac{P}{K}\right).$$

The effect of the new factor $1 - \frac{P}{K}$ depends on the population level with respect to K . The effects are summarized in the table below. We call

$$P' = rP \left(1 - \frac{P}{K}\right)$$

the **logistic growth model**. It is a more realistic extension of the basic growth model because the growth rate now depends on the size of the population relative to its carrying capacity. Since this is an autonomous equation, we can analyze the population behavior without explicitly solving the DE.

Population	$1 - \frac{P}{K}$	Sign of P'	Effect
$P \ll K$	≈ 1	+	nearly basic growth $P' \approx rP$
$0 < P < K$	> 0	+	population grows
$P = K$	0	0	no population growth
$P > K$	< 0	-	population declines

Figure 1: We consider four different population levels. As population $P(t)$ grows from very small to above K , the sign of P' changes from positive to zero to negative.

EXAMPLE 2

A fish population follows logistic growth with an intrinsic growth rate $r = 0.4$ per year and a carrying capacity of $K = 10000$ fish. Suppose the initial population of fish is $P(0) = 2000$. What happens to the population in the long run? What if the initial population is 12000?

Solution.

Set up the model. This is a logistic model with $r = 0.4$ per year and $K = 10000$ fish. The initial condition is $P(0) = 2000$ fish. We may write the model as the following IVP:

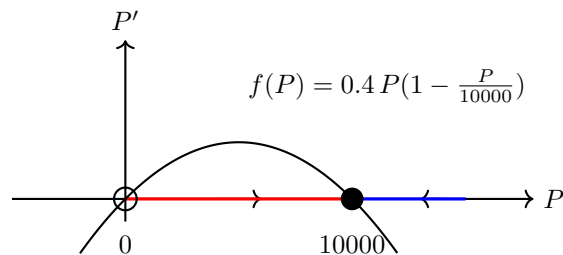
$$\begin{cases} P' = 0.4 P \left(1 - \frac{P}{10000}\right) \\ P(0) = 2000 \end{cases}$$

We can solve this equation, but we will instead use the graphical methods that we learned on Day 06.

Find the equilibria and their stability The equilibria can be found by setting the RHS equal to zero and solving for P :

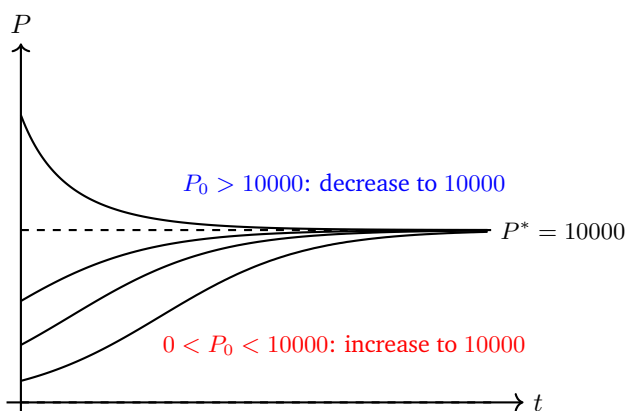
$$\begin{aligned} f(P) &= 0.4 P \left(1 - \frac{P}{10000}\right) \\ 0 &= 0.4 P \left(1 - \frac{P}{10000}\right) \\ \implies P^* &= 0, \quad P^* = 10000 \end{aligned}$$

The P' versus P graph and phase line are shown below. Since negative populations are not physically meaningful, we only consider $P \geq 0$.



The phase line shows that $P^* = 0$ is unstable and $P^* = 10000$ is stable.

Interpret the solutions. If $P(0) = 2000$ fish, then $0 < P_0 < 10000$, so $P' > 0$ and the population increases toward 10000. If $P(0) = 12000$, then $P_0 > 10000$, so $P' < 0$ and the population decreases toward 10000. The solution curves are plotted below.



Thus, in either case,

$$\lim_{t \rightarrow \infty} P(t) = 10000.$$

The fish population stabilizes at 10,000 fish regardless of the starting condition. The carrying capacity is therefore a sustainable population level, not a hard upper bound.

3. Harvesting

3.1 Basic Growth with Harvesting

Recall that the basic growth model is

$$P' = rP,$$

where r is the intrinsic growth rate. The term rP represents *percentage-based growth*: as the population increases, the number of individuals added per unit time also increases.

Now suppose that individuals are harvested from the population at a *constant rate* $h > 0$. Unlike rP , the harvesting rate does not depend on the size of the population. It is a *flat rate*. For example, harvesting $h = 100$ fish per month means that 100 fish are removed each month, regardless of the current fish population. Combining intrinsic growth with constant harvesting gives

$$P' = rP - h.$$

The parameter is called the **harvesting rate**, and it has units of population per unit time. Let's practice setting up a model.

EXAMPLE 3

A deer population has an intrinsic growth rate of 12% per year. Hunters remove deer from the population at a rate of 5 deer per month. Write a differential equation model for the deer population, where t is measured in years.

Solution.

Since t is measured in years, both rates must be expressed in terms of years. The intrinsic growth rate is

$$r = 0.12 \text{ per year.}$$

The harvesting rate is

$$h = 5 \frac{\text{deer}}{\text{month}} \left(\frac{12 \text{ months}}{1 \text{ year}} \right) = 60 \frac{\text{deer}}{\text{year}}.$$

Therefore, the model is

$$P' = 0.12P - 60.$$

3.2 Critical Population Threshold

Consider the basic growth model with constant harvesting:

$$P' = rP - h, \quad r, h > 0.$$

This autonomous equation has one equilibrium

$$P^* = \frac{h}{r}.$$

To determine its stability, consider the function $f(P) = rP - h$. We can plot $f(P)$ on the P' versus P graph. This is an increasing linear function of P that crosses the P -axis at $P = h/r$. We can draw the phase line directly on the P -axis.

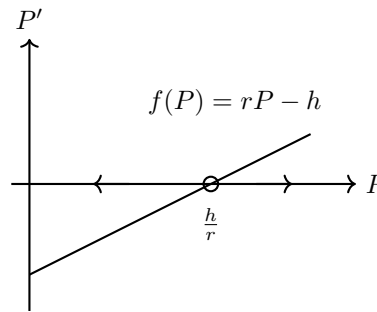


Figure 2: P' versus P for the basic growth model with constant harvesting.

The arrows point away from $P^* = h/r$, so the equilibrium is unstable. Depending on whether the initial population $P_0 = P(0)$ is below or above the equilibrium, the behavior of the population changes.

- If $0 < P_0 < \frac{h}{r}$, then $P' < 0$ and the population decreases.
- If $P_0 > \frac{h}{r}$, then $P' > 0$ and the population increases.

Thus, the equilibrium $P^* = \frac{h}{r}$ acts as a **critical population threshold**. Below this threshold, harvesting outpaces intrinsic population growth. Above this threshold, intrinsic population growth outpaces harvesting.

3.3 Logistic Growth with Harvesting

We can also add constant harvesting to the logistic growth model:

$$P' = rP \left(1 - \frac{P}{K} \right) - h.$$

This is still an autonomous DE, so we can analyze its behavior using the P' versus P graph and phase line. Recall that

$$f(P) = rP \left(1 - \frac{P}{K} \right)$$

is a downward-opening parabola. Subtracting the constant harvesting rate h simply shifts this parabola downward by h units. If the shifted parabola crosses the P -axis twice, there are two equilibria $P_-^* < P_+^*$. From the phase line, P_-^* is unstable and P_+^* is stable. For the exact values of the equilibria, you will need the quadratic formula.

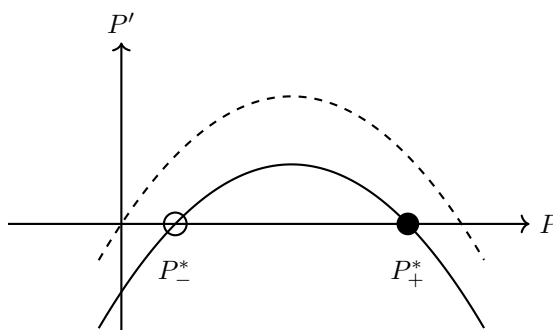


Figure 3: This is the P' vs. P graph for the logistic model with harvesting. The left equilibrium P_-^* is unstable and the right P_+^* is stable

4. Fixed Population Models

So far, our population models have described how the total size of a population changes. We can also model how individuals *move between groups* within a population.

4.1 Two Groups

Suppose a population of N individuals is divided into two groups:

$$A(t) = \text{number of individuals in group A}, \quad B(t) = \text{number of individuals in group B}.$$

Although $A(t)$ and $B(t)$ may change over time, we assume that the total population is fixed:

$$A(t) + B(t) = N.$$

Individuals may move between the two groups. For example, an individual from group B can leave and enter group A . Thus, as one group gains individuals, the other group loses them.

4.2 Goal

Our goal is to develop a model that describes how the two groups change over time. In other words, we want to develop differential equations for

$$A'(t) \quad \text{and} \quad B'(t).$$

The equations will depend on how individuals move between groups A and B .

EXAMPLE 4

A company has 1000 customers divided between two subscription plans: Basic and Premium. Each month, 10% of Basic customers upgrade to Premium, while 5% of Premium customers downgrade to Basic. Develop a fixed population model for the number of Basic and Premium customers.

Solution.

Let $B(t)$ and $P(t)$ be the number of Basic and Premium customers, respectively. Since the total number of customers is fixed, we let $B(t) + P(t) = 1000$.

Each month, 10% of the Basic customers upgrade to Premium. Therefore, B decreases at a rate of $0.10B$, while P increases at the same rate:

$$B' = -0.10B + \dots, \quad P' = 0.10B + \dots$$

Similarly, 5% of the Premium customers downgrade to Basic. Therefore, P decreases at a rate of $0.05P$, while B increases at the same rate:

$$B' = -0.10B + 0.05P, \quad P' = 0.10B - 0.05P.$$

Therefore, the model is

$$\begin{cases} B' = -0.10B + 0.05P, \\ P' = 0.10B - 0.05P. \end{cases}$$

4.3 Fixed Population Constraint

In a fixed population model,

$$A(t) + B(t) = N.$$

Therefore, the two populations are not independent. If we know one population, we automatically know the other:

$$B(t) = N - A(t).$$

There is also a corresponding relationship between the rates of change. Differentiating $A + B = N$ gives

$$A'(t) + B'(t) = 0 \implies A' = -B'.$$

For instance, in Example 3, we had $B' = -0.10B + 0.05P = -P'$. This means that a model with two subpopulations can be reduced to a single differential equation. In the previous example, the number of Premium customers is

$$P(t) = 1000 - B(t).$$

Substituting P into the equation for B' yields

$$B' = -0.10B + 0.05P = -0.10B + 0.05(1000 - B) = 500 - 0.15B.$$

This new DE has the form

$$B' = f(B)$$

which is an autonomous equation.

4.4 Interaction Terms

Sometimes the rate at which individuals change groups depends on *interactions* between the two groups. Suppose an individual from group A must interact with an individual from group B before either can change groups. The frequency of these interactions is proportional to the product

$$AB.$$

We, therefore, model the interaction rate by

$$kAB,$$

where $k > 0$ is an **interaction rate**. The larger either group becomes, the more opportunities there are for individuals from the two groups to interact.

SIR Model

Fixed-population models are frequently used in epidemiology. Scientists use **SIR model** to study a population of people with an infectious disease. A common example is a flu. Here are the groups:

- $S(t)$ = number of people that are susceptible
- $I(t)$ = number of people that are infected
- $R(t)$ = number of people that have recovered and developed immunity

The sum of each subpopulation must add up to a total of N at all times. A sick person may randomly encounter a healthy person. When this happens, the sick person transmits the disease to the healthy person. A sick person can eventually recover with their immune system. When a person overcomes the disease, they gain immunity from it so they do not catch the disease again. A simplified model would be

$$\begin{cases} S' &= -kIS, \\ I' &= kIS - rI, \\ R' &= rI, \end{cases}$$

where k and r are positive constants. The product kIS represents the encounter between a sick person and a healthy person. The parameter r represents how quickly a person overcomes the flu. There are many variants of this model. For example, there could be a deceased group or a vaccinated group. We may have a group of carriers that don't show any symptoms.