

1.4 – Linear Differential Equations: Integrating Factors

(Section 1.4.1)

1. Linear vs. Nonlinear

1.1 Definitions

The most general first-order differential equation (DE) has the form

$$x' = f(t, x),$$

where $f(t, x)$ is some combination of t and x . Some can be written in the form

$$x' + p(t)x = q(t),$$

where $p(t)$ and $q(t)$ are functions of t only. This equation arrangement is called **normal form**. If a DE can be written in normal form, then it is said to be **linear**. Otherwise, the DE is **nonlinear**.

1.2 Linear Examples

Each of the following equations is linear. Some equations require manipulation to be put into normal form.

Equation	Normal Form
$x' + 3x = t$	Already normal
$x' = tx$	$x' - tx = 0$
$(\sin t)x' = 3tx + \sqrt{t}$	$x' - \frac{3t}{\sin t}x = \frac{\sqrt{t}}{\sin t}$
$t^2x' - 4(t+1) = 0$	$x' = \frac{4(t+1)}{t^2}$

1.3 Nonlinear Examples

Each of the following equations is nonlinear because x and x' appear in a nonlinear combination.

Equation	Why Nonlinear?
$x' = x^2$	x is squared
$x' + \sin(x) = t$	$\sin x$ is nonlinear
$xx' + t = 0$	product of x and x'
$(x')^2 + x = e^t$	x' is squared

1.4 Goal

The goal of this lecture is to learn a technique to solve

$$x' + p(t)x = q(t)$$

using *integrating factors*.

2. Solving a Linear DE

2.1 Linear Example

Suppose you want to solve

$$x' + \frac{2}{t}x = 3e^{t^3}.$$

The antiderivative technique does not work. Separation of variables does not work. Notice that it is a linear DE. Something interesting occurs when we multiply both sides by $\mu(t) = t^2$:

$$\begin{aligned}x' + \frac{2}{t}x &= 3e^{t^3} \\ \mu(x' + \frac{2}{t}x) &= \mu(3e^{t^3}) && \text{(mult. by } \mu) \\ t^2(x' + \frac{2}{t}x) &= 3t^2e^{t^3} \\ t^2x' + 2tx &= 3t^2e^{t^3}. && \text{(distribute)}\end{aligned}$$

There is something special about the LHS that is related to the product rule for derivatives.

2.2 Total Derivative

Let $f(t) = t^2$ and $x = x(t)$. Notice that the LHS of the equation is the result of a product rule for derivatives:

$$\begin{aligned}\frac{d}{dt}(f(t)x(t)) &= f'x + fx' \\ &= (t^2)(x)' + (t^2)'(x) \\ &= t^2x' + 2tx.\end{aligned}$$

The compact form of the product rule is called the **total derivative**:

$$t^2x' + 2tx = \underbrace{(t^2x)'}_{\text{total derivative}}.$$

There is a special advantage to rewriting it this way.

2.3 Advantage

The advantage of rewriting the LHS as the total derivative is that we can now integrate both sides with respect to t , and solve for x :

$$\begin{aligned}t^2x' + 2tx &= 3t^2e^{t^3} \\ (t^2x)' &= 3t^2e^{t^3} && \text{(total deriv.)} \\ \int (t^2x)' dt &= \int 3t^2e^{t^3} dt && \text{(integrate)} \\ t^2x &= e^{t^3} + C \\ x &= t^{-2}e^{t^3} + Ct^{-2} && \text{(solve for } x)\end{aligned}$$

We have solved a seemingly difficult DE simply by multiplying both sides by t^2 .

2.4 Summary

Let's summarize the steps:

$$\begin{aligned}x' + \frac{2}{t}x &= 3e^{t^3} \\ t^2x' + 2tx &= 3t^2e^{t^3} && \text{(1. Mult. by } t^2) \\ (t^2x)' &= 3t^2e^{t^3} && \text{(2. Total Deriv.)} \\ t^2x &= e^{t^3} + C && \text{(3. Integrate)} \\ x &= t^{-2}e^{t^3} + Ct^{-2} && \text{(4. Divide } t^2)\end{aligned}$$

We have arrived at our solution. Multiplying by $\mu = t^2$ was the key to unraveling the entire problem. We call μ the *integrating factor*. We will discuss how to find μ next.

3. General Theory

3.1 Integrating Factor

To solve the linear DE

$$x' + p(t)x = q(t),$$

multiply both sides of the equation by the **integrating factor** $\mu(t)$ so that the LHS becomes a total derivative:

$$\begin{aligned} x' + p x &= q \\ \mu(x' + p x) &= \mu q \\ \mu x' + \mu p x &= \mu q && \text{(line 3)} \\ \underbrace{(\mu x)'}_{\text{total deriv.}} &= \mu q && \text{(line 4)} \end{aligned}$$

By comparing the left sides of line 3 and line 4, we can derive an equation for μ . That means μ must satisfy

$$\begin{aligned} \mu x' + \mu p x &= (\mu x)' \\ \mu x' + \mu p x &= \mu' x + \mu x' && \text{(product rule)} \\ \cancel{\mu x'} + \mu p x &= \mu' x + \cancel{\mu x'} && \text{(cancel common term)} \\ \mu p x &= \mu' x && \text{(divide by } x) \end{aligned}$$

Finally, we get the simplified equation for μ

$$\mu' = \mu p.$$

3.2 Integrating Factor Formula

We may obtain the integrating factor $\mu(t)$ by solving

$$\frac{d\mu}{dt} = \mu p.$$

This can be done using separation of variables

$$\begin{aligned} \frac{d\mu}{\mu} &= p dt \\ \ln |\mu| &= \int p dt + C \\ \mu &= C e^{\int p dt} \end{aligned}$$

Although there is an arbitrary constant in μ , it is not needed when solving linear DEs. This is because we multiply both sides of the equation by μ . Therefore, the formula for the integrating factor is

$$\mu(t) = e^{\int p(t) dt}.$$

3.3 How to Solve Linear DEs

Now let's go over the general procedure for solving linear DEs.

1. Put the DE in normal form.

$$x' + p(t)x = q(t)$$

2. Calculate the integrating factor.

$$\mu = e^{\int p(t) dt}$$

3. Multiply both sides by μ , and then rewrite the LHS as the total derivative.

$$\begin{aligned}\mu(x' + p(t)x) &= \mu q(t) \\ (\mu x)' &= \mu q(t).\end{aligned}$$

4. Integrate both sides. Be sure to add C .

$$\mu x = \int \mu q(t) dt + C$$

5. Divide both sides by μ .

$$x = \frac{1}{\mu} \int \mu q(t) dt + C \frac{1}{\mu}$$

4. Worked Examples

For the rest of the lecture, we'll work through complete examples.

EXAMPLE 1

Find the general solution to the DE using the method of integrating factors

$$x' = e^{-t} - 2x.$$

Solution.

Step 1: Put DE in normal form.

$$\begin{aligned}x' &= e^{-t} - 2x \\ x' + 2x &= e^{-t}\end{aligned}$$

Step 2: Find the integrating factor.

$$\begin{aligned}\mu &= e^{\int p dt} \\ &= e^{\int 2 dt} \\ &= e^{2t}.\end{aligned}$$

Step 3: Multiply both sides by μ and rewrite the LHS.

$$\begin{aligned}e^{2t}(x' + 2x) &= e^{2t}e^{-t} \\ (e^{2t}x)' &= e^t\end{aligned}$$

Step 4: Integrate both sides.

$$\begin{aligned}\int (e^{2t}x)' dt &= \int e^t dt \\ e^{2t}x &= e^t + C.\end{aligned}$$

Step 5: Divide by μ .

$$\begin{aligned}x &= \frac{e^t + C}{e^{2t}} \\ &= e^{-t} + Ce^{-2t}\end{aligned}$$

The general solution is

$$x(t) = e^{-t} + Ce^{-2t}.$$

The next example involves an initial condition.

EXAMPLE 2

Find the particular solution to the IVP

$$\begin{cases} tx' = x + t^3, \\ x(1) = 0. \end{cases}$$

Solution.

Step 1: Put DE in normal form.

$$\begin{aligned} tx' &= x + t^3 \\ tx' - x &= +t^3 \\ x' + \underbrace{\left(-\frac{1}{t}\right)}_{p(t)} x &= t^2. \end{aligned}$$

Step 2: Find the integrating factor.

$$\begin{aligned} \mu &= e^{\int p \, dt} \\ &= e^{\int -\frac{1}{t} \, dt} \\ &= e^{-\ln t} = e^{\ln \frac{1}{t}} = \frac{1}{t}. \end{aligned}$$

Don't include $+C$ when calculating μ .

Step 2: Multiply the equation by μ , and rewrite the LHS.

$$\begin{aligned} \frac{1}{t}(x' - \frac{1}{t}x) &= \frac{1}{t}(t^2) \\ \frac{d}{dt}\left(\frac{x}{t}\right) &= t \end{aligned}$$

Step 3: Integrate both sides.

$$\begin{aligned} \int \frac{d}{dt}\left(\frac{x}{t}\right) dt &= \int t \, dt \\ \frac{x}{t} &= \frac{1}{2}t^2 + C. \end{aligned}$$

This is the step where you include $+C$.

Step 4: Solve for $x(t)$.

$$x(t) = \frac{1}{2}t^3 + Ct.$$

Step 5: Apply the initial condition. Using $x(1) = 0$,

$$0 = \frac{1}{2} + C \quad \Rightarrow \quad C = -\frac{1}{2}.$$

$$\boxed{x(t) = \frac{1}{2}(t^3 - t).}$$

EXAMPLE 3

Find the general solution to the DE

$$x' = 2e^{-t} - 3x.$$

Solution.

Step 1: Put the DE in normal form.

$$\begin{aligned} x' &= 2e^{-t} - 3x \\ x' + \underbrace{(3)}_p x &= 2e^{-t}. \end{aligned}$$

Step 2: Compute the integrating factor.

$$\mu(t) = e^{\int p(t) dt} = e^{\int 3 dt} = e^{3t}.$$

Step 3: Multiply the DE by $\mu(t)$, then rewrite the LHS as the total derivative.

$$\begin{aligned} x' + 3x &= 2e^{-t} \\ e^{3t}(x' + 3x) &= 2e^{-t}e^{3t} \\ (e^{3t}x)' &= 2e^{2t}. \end{aligned}$$

Step 4: Integrate both sides.

$$\begin{aligned} \int (e^{3t}x)' dt &= \int 2e^{2t} dt \\ e^{3t}x &= e^{2t} + C \end{aligned}$$

Step 5: Solve for x .

$$\begin{aligned} e^{3t}x &= e^{2t} + C \\ x &= e^{-t} + Ce^{-3t}. \end{aligned}$$

The general solution is

$x(t) = e^{-t} + Ce^{-3t}.$

4.1 Warning About the Formula

There is a one-shot formula to solve

$$x' + p(t)x = q(t).$$

The general solution is

$$x(t) = e^{-P(t)} \left(\int e^{P(t)} q(t) dt + C \right),$$

where $P(t)$ is the antiderivative of $p(t)$. There is nothing new in this formula. It is the integrating factor method unrolled into one big equation. I have seen many students try to replicate this formula poorly on exams. I want you to know that this exists, but I do not recommend memorizing it. It is far better to understand the integrating factor method.