

1.2 – Growth/Decay Model, Solution Curves, and Direction Fields

(Sections 1.1.2, 1.1.3)

1. Verifying a Solution to a DE

1.1 Procedure

To show that a function $x(t)$ solves a differential equation (DE), you must show that the left-hand side (LHS) equals the right-hand side (RHS) of the equation. If you are given an initial condition (IC), you must verify the equation as well. Here is an example.

EXAMPLE 1

Suppose k is a constant. Show that the function $x(t) = x_0 e^{kt}$ is a solution to the initial value problem

$$\begin{cases} x' = kx \\ x(0) = x_0 \end{cases}.$$

Solution.

Step 1: First, substitute x into the LHS

$$\begin{aligned} \text{LHS} &= x' \\ &= (x_0 e^{kt})' \\ &= kx_0 e^{kt}. \end{aligned}$$

Step 2: Next, substitute x into the RHS and show that it is identical to the LHS

$$\begin{aligned} \text{RHS} &= kx \\ &= k(x_0 e^{kt}) \\ &= \text{LHS} \quad \checkmark \end{aligned}$$

Step 3: Lastly, verify the initial condition

$$x(0) = x_0 e^0 = x_0 \quad \checkmark$$

1.2 Formula

The IVP $x' = kx$ with $x(0) = x_0$ is an important equation that will arise frequently in this class. In the last example, we have established that

$$x = x_0 e^{kt}$$

is the solution. You should get into the habit of using this result as a *formula*.

EXAMPLE 2

Find the solution to the following DEs/IVPs.

- a) $x' = 7x$
- b) $x' = -3x, x(0) = 10.$

Solution.

- a) The coefficient is $k = 7$. This quantity becomes the coefficient in the exponent of the solution. The coefficient in front can just be C . The general solution is

$$x(t) = Ce^{7t}$$

- b) The coefficient is $k = -3$, so the general solution is

$$x(t) = Ce^{-3t}.$$

We can apply the general solution and solve for C . That will give us the solution to the IVP:

$$x(0) = 10 = C, \implies x(t) = 10e^{-3t}$$

2. Types of Solutions

2.1 General Solution

Consider the DE

$$x' = -kx$$

for $k > 0$. Based on our formula, we know

$$x(t) = Ce^{-kt}$$

solves the equation. We call this the **general solution** because this one function captures every solution using an arbitrary constant C .

2.2 Family of Solutions

This function $x(t) = Ce^{-kt}$ also captures discrete solutions for each value of C . For instance, we can let $C = 1, 0.5, -0.5, -1$, etc. This creates an infinite set of functions that are all solutions:

$$\mathcal{F} = \{\dots, -e^{-kt}, -0.5e^{-kt}, 0.5e^{-kt}, e^{-kt} \dots\}$$

We refer to this set $\mathcal{F}$ as a **family of solutions**. We can plot a few members of the family in one graph to visualize them. Here is an interesting fact: **a family of solutions can never intersect.**

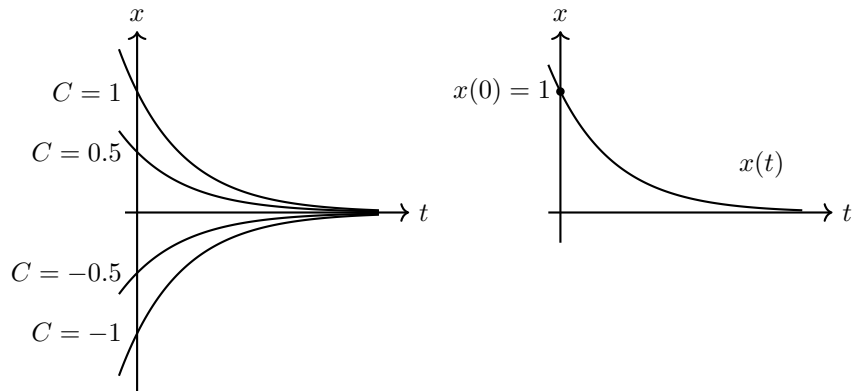


Figure 1: The left figure shows a family of solutions to $x' = -kx$. The right figure shows the one particular solution corresponding to an initial condition $x(0) = 1$.

2.3 Particular Solution

Consider the initial value problem (IVP)

$$\begin{cases} x' = -kx, \\ x(0) = 1. \end{cases}$$

Using the initial condition (IC) $x(0) = 1$, we may solve for the value of C :

$$1 = Ce^{-k(0)} = C,$$

so the solution is

$$x(t) = e^{-kt}.$$

We call this a **particular solution** of $x' = -kx$ because it satisfies a particular IC. The IC $x(0) = 1$ corresponds to the point $(0, 1)$ on the tx -plane. Fact: **the particular solution is the one member of the family passing through the point $(0, 1)$.**

3. The Growth/Decay Model

This section is about equations of the form $x' = kx$.

3.1 Growth Model

Suppose $x(t)$ represents the quantity of something at time t . Then the derivative x' represents its rate of change. Suppose that the rate of change of $x(t)$ is proportional to the amount $x(t)$ itself:

$$x' \propto x.$$

Introducing a constant of proportionality $k > 0$, we write

$$x' = kx.$$

This equation is called the **growth model**. If x is small, then x' is also small. If x is large, then x' is also large. In other words, the more you have, the faster the quantity grows.

3.2 Exponential Growth

The model $x' = kx$ is appropriate for:

- $x(t)$ = amount of money in a bank account earning compound interest,
- $x(t)$ = population of bacteria reproducing freely,
- $x(t)$ = the size of a snowball rolling down a snowy hill.

In each case, the quantity $x(t)$ grows slowly at first, but then speeds up as the $x(t)$ itself becomes large. This leads to rapid accumulation due to a snowball effect. The way $x(t)$ increases with respect to time is called **exponential growth**.

3.3 Word Problem

We can use this knowledge about the growth model to answer some questions. The steps are:

1. Set up the model.
2. Solve the IVP.
3. Use the solution to answer the question.

Here is an example.

EXAMPLE 3

Suppose \$1000 is deposited into a bank account that earns interest continuously at an annual rate of 5%. Determine how long it takes for the balance to double.

Solution.

1. **Set up the model.** Let $x(t)$ be the balance in dollars after t years. The initial balance is \$1000 so we let $x(0) = 1000$. Since the growth rate is $r = 5\% = 0.05$, our model is

$$\begin{cases} x' = 0.05x, \\ x(0) = 1000. \end{cases}$$

2. **Solve the IVP.** Any DE of the form $x' = kx$ always has the same form. We may write the solution using the formula

$$x(t) = x_0 e^{kt} = 1000 e^{0.05t}.$$

3. **Use the solution.** Set $x(t) = 2000$ and solve for t :

$$2000 = 1000 e^{0.05t}$$

$$2 = e^{0.05t}$$

$$\ln 2 = 0.05t$$

$$t = \frac{\ln 2}{0.05} \approx 13.86.$$

Final answer: it takes $t \approx 13.9$ years for the bank balance to grow from \$1000 to \$2000.

3.4 Decay Model

Again, let's suppose $x(t)$ represents the amount of some material at time t . If the derivative is negative ($x' < 0$), then that means the amount is *decreasing* over time. If the rate of decrease is proportional to the remaining amount, then we write

$$x' \propto -x,$$

or rather

$$x' = -kx,$$

where $k > 0$. The parameter k represents the percentage amount lost after each time step. This differential equation is called the **decay model**. This DE can model:

- radioactive decay of a substance,
- cooling of a hot object in a cooler environment,
- discharge of a capacitor.

In the decay model, the quantity decreases quickly at first, but the rate of decay slows as less material remains.

Here is a quick example. Let's say a rubber ball bounces back up 50% of its height from the previous bounce. If it's dropped from 1 meter, then it bounces up to 0.5 meters, 0.25 meters, etc. If x is the bounce height and t is the number of bounces, we may write

$$x' = -0.5x, \quad x(0) = 1$$

to model the ball.

4. Geometric Approach

We may gain insight into the solution of a first-order DE without explicitly solving the equation. In this section, we focus on understanding the qualitative behavior of solution curves from a geometric perspective.

4.1 Interpreting IVPs

Consider the first-order IVP

$$\frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0,$$

where the goal is to find the unknown function $x(t)$. Let's begin by establishing a mental picture of the solution. You can interpret the solution as a curve existing in the **tx-plane**. We refer to the plotted function as the **solution curve** or **integral curve**.

There are three important aspects of IVPs:

1. **Slope** $\frac{dx}{dt}$: the slope tells us the "steepness" of the line tangent to the solution curve.
2. **Formula** $f(t, x)$: the formula is effectively a function that allows us to calculate the slope of the solution at any point in the (t, x) plane.
3. **Initial Condition** $x(t_0) = x_0$: the IC tells us that the curve must pass through the point (t_0, x_0) .

The figure below illustrates these ideas geometrically. We may obtain a rough idea of the solution curve without solving anything.

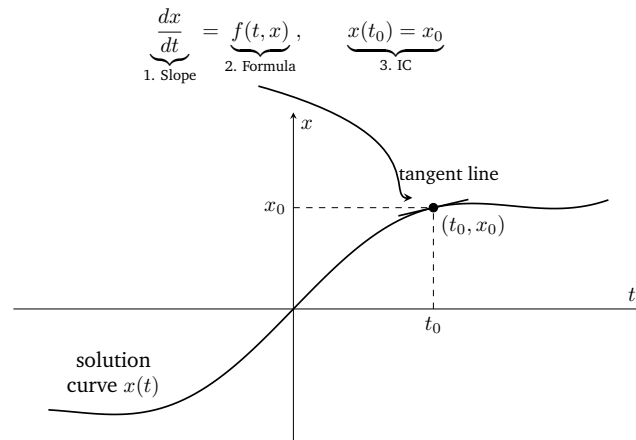


Figure 2: Think of first-order IVPs as a task to find a solution curve $x(t)$ going through the point (t_0, x_0) which obeys a specified slope formula $f(t, x)$.

4.2 Slope Analysis

There are two interesting slopes: $x' = 0$ and $x' = \pm\infty$. If a certain combination of (t, x) forces $x' = f(t, x)$ to be zero, then we know there's a *horizontal tangent* located at (t, x) . If the slope is $\pm\infty$ (in the limit), then there must be a *vertical tangent*. Let's see an example.

EXAMPLE 4

Consider the IVP

$$x' = -x^2, \quad x(2) = 1.$$

What can you infer about the solution curve? Assume $x > 0$.

Solution.

First of all, we know the solution goes through the point $(2, 1)$. Next, notice that since

$$x' = -x^2 \leq 0$$

for every value of x , the curve must be *monotonically decreasing*. At the point $(2, 1)$, the slope is

$$x' = -(1)^2 = -1.$$

Thus, the tangent line at $x = 1$ has a slope of -1 .

Next, let us examine how the slope behaves for large values of x . Since

$$\lim_{x \rightarrow \infty} -x^2 = -\infty,$$

the slope becomes extremely negative when x is large. As we move north from $(2, 1)$, the curve becomes a vertical asymptote. Since

$$\lim_{x \rightarrow 0^+} -x^2 = 0,$$

the curve should flatten out as we move south from $(2, 1)$ towards the t -axis. The general solution is

$$x(t) = \frac{1}{t + C},$$

and is plotted below. The function curve of the function is as predicted.

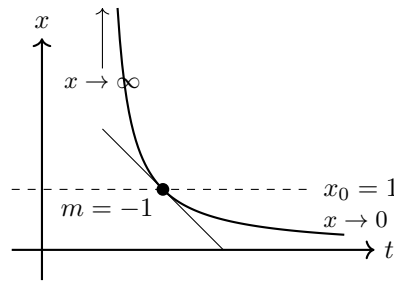


Figure 3: The general solution to $x' = -x^2$ is $x(t) = \frac{1}{t+C}$. The plot above shows the particular solution $x(t) = \frac{1}{t-1}$.

4.3 Direction Fields

In the DE

$$x' = f(t, x)$$

the function $f(t, x)$ gives us a formula for the slope at any point (t, x) . We may populate an entire tx -plane full of slopes calculated at their respective coordinates. The resulting picture is called the **direction field**.

For instance, consider the DE

$$x' = f(t, x) = t^2 - x.$$

Let's choose the point $(-1, 2)$ ($t = -1$ and $x = 2$). Since $f(-1, 2) = (-1)^2 - 2 = -1$, we draw a line with slope $m = -1$ at the coordinate $(-1, 2)$. The figure below shows the direction field.

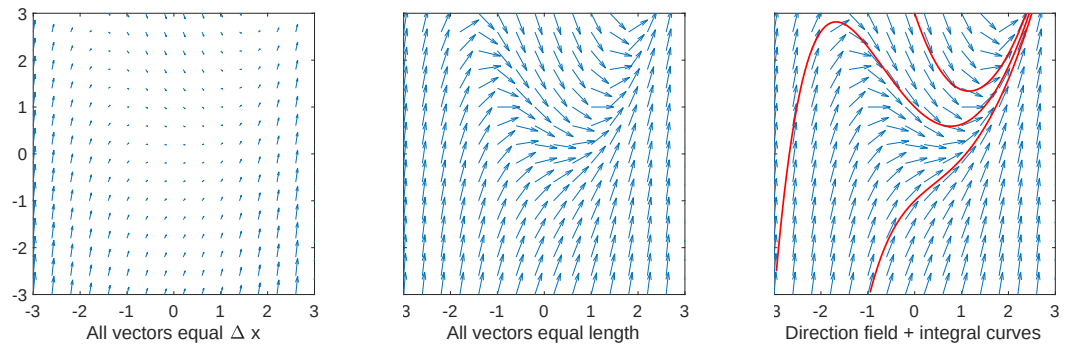


Figure 4: The figures above show direction fields for the equation $x' = t^2 - x$.

4.4 Solution Curves on Direction Fields

What makes direction fields so powerful? **They let us sketch solutions to IVPs for any IC.** The figure below shows the direction field for the IVP

$$x' = t + \sin(x), \quad x(t_0) = x_0.$$

The red curves are all solutions to the DE corresponding to different initial conditions. To obtain the solution curve,

1. draw a dot at the point (t_0, x_0) .
2. trace a path in the direction of the slope
3. stop when you encounter a new slope
4. return to step 2

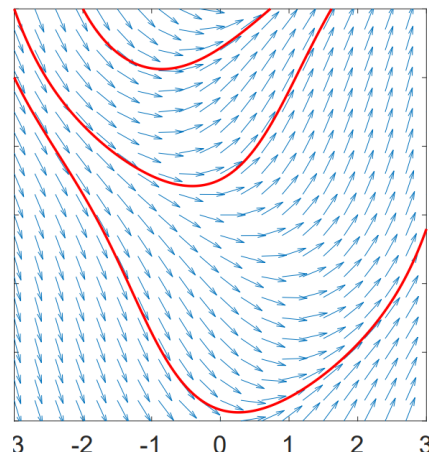


Figure 5: These figures show how solution curves can be drawn on direction fields. This is example 1 on the syllabus page <https://math.unm.edu/~nitsche/courses/316/>. You are encouraged to see the other five examples.

5. Graphing by Superposition

This section is about plotting sums of functions such as $y_1 = x^2$ plus $y_2 = \cos x$.

5.1 Common Functions

You are expected to know how to sketch the following common functions by hand:

- $y = k$ (constant)
- $y = x$
- $y = x^2$
- $y = x^3$
- $y = \sqrt{x}$
- $y = 1/x$
- $y = 1/x^2$
- $y = e^x$
- $y = e^{-x}$
- $y = \cos(x)$
- $y = \sin(x)$

These functions are less common, but they are relevant to this course

- $y = \frac{1}{1+e^{-x}}$ (S-shaped sigmoid function)
- $y = \tan^{-1}(x)$ (looks like an S with asymptotes at $\pm\pi/2$)
- $y = \cosh(x)$ (looks like $x^2 + 1$)
- $y = \sinh(x)$ (looks like x^3)

5.2 Graphing Technique

Graphing by superposition is a technique that allows you to sketch the sum of two functions such as $f = 1 + \sin x$. The idea is to sketch the two pieces of the function together on one graph. Then you sketch the sum by combining the elevations of the two functions. When you add two positive functions, the sum is shifted upwards. When you add two negative functions, the sum is shifted downwards.

EXAMPLE 5

Sketch $y = x + \sin x$ by hand.

Solution.

Begin by sketching $y_1 = x$ and $y_2 = \sin x$ in dashed lines on the same graph. Then plot the sum by combining the heights of both functions. Since $y = x$ is negative for $x < 0$, the sine wave is shifted down. Similarly, $y = x$ is positive for $x > 0$, so the sine wave is shifted up.

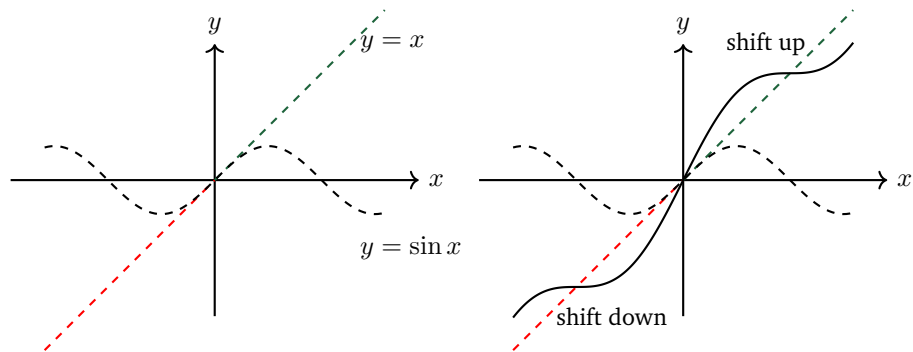


Figure 6: Graphing by superposition: the dashed curves are $y = x$ and $y = \sin x$, and the solid curve is their sum $y = x + \sin x$.